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Stability Analysis of a Lotka-Volterra Type Biological Model in Its Fractional Version

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Abstract: The stability of systems of ordinary differential equations of fractional order has been a study subject over the last two decades within fractional analysis. This study presented some fundamental results on the stability of linear fractional systems, which has allowed, in a certain sense, the generalization of some results for nonlinear fractional differential equations. This paper aims to present the fundamental results of the stability of systems of nonlinear ordinary differential equations of fractional order. These results are specified and demonstrated for a particular type of nonlinear differential equations of fractional order and applied to the analysis of the dynamics of a Lotka–Volterra-type biological model, which allows an analysis of the interaction between three species, some acting as prey and others as predator. The stability analysis of the prey–predator biological model used fractional operators in the sense of Caputo, and simulations verified these results.

Keywords: fractional calculus, fractional differential equations, stability of fractional differential equations, biological models, Lotka-Volterra type model.

分數階洛特卡-沃泰拉型生物模型的穩定性分析

摘要：分數階常微分方程組的穩定性在過去二十年一直是分數階分析的研究主題。這項研究提出了關於線性分數階系統穩定性的一些基本結果，在某種意義上允許推廣非線性分數階微分方程的一些結果。本文旨在介紹分數階非線性常微分方程組穩定性的基本結果。這些結果針對特定類型的分數階非線性微分方程式進行了指定和論證，並應用於洛特卡沃泰拉型生物模型的動力學分析，該模型允許分析三個物種之間的相互作用，其中一些物種充當獵物和其他人作為掠奪者。捕食者生物模型的穩定性分析使用了卡普托意義上的分數算子，模擬驗證了這些結果。

关键词：分數階微積分、分數階微分方程、分數階微分方程的穩定性、生物模型、洛特卡-沃泰拉型模型。

1. Introduction

Studying sensitive variations in the initial conditions is fundamental in the qualitative analysis of differential equations. This intuitive idea makes it possible to define stability for systems of ordinary differential equations, which are the object of study in mathematical models applied to various disciplines of the exact sciences, where the aim is to provide necessary and sufficient conditions so that, in a given problem with established initial conditions, solutions close to this initial value remain close during all future time or, in a given case, tend to the equilibrium solution [1]. The Russian physicist Lyapunov laid the foundations for this concept of proximity in his doctoral thesis "The General Problem of Motion Stability" [2], in which he presented two methods to establish the stability of systems of ordinary differential equations.

The fundamental work of Lyapunov, which establishes the stability of systems of ordinary differential equations in the classical sense, has been generalized to fractional calculus. Not all the stability results have been extended to the non-integer case, that is, to fractional differential equations, but significant results established the stability of systems of ordinary differential equations of fractional order, both linear and nonlinear.

The significant results obtained from the modeling and analysis of problems applied to science and engineering through fractional operators are due to the non-local character of their fractional operators, which allows many of the physical phenomena with special memory and genetic characteristics modeled by ordinary differential equations of fractional order [3-14].

Although fractional order differential equations have attracted the attention of many researchers, only a few have made significant contributions to the construction of a clear and coherent theory of these equations addressed in the qualitative approach, which is an open research topic and has not had a significant breakthrough leading to the analysis of stability concepts on the dynamics of the solutions of such systems. There is a gap in structured stability theory for this type of differential equation; however, some researchers have presented results from different contexts and focused on stability for fractional ordinary differential equations of linear and nonlinear type.

Although, as mentioned above, there is still no established theory of the stability of differential equations with fundamental contributions to this theory made in the last two decades. Dennis Matignon is one of the pioneers in establishing stability results for systems of linear fractional order ordinary differential equations. He analyzed for the first time the stability of systems of finite dimensional fractional linear differential equations applied to control theory. He mentioned that stabilities are guaranteed if the roots of some polynomial lie outside $|\arg(\sigma)| \leq \alpha\pi/2$, thus generalizing the known results to the ordinary case $\alpha=1$ [15]. From that time, several researchers became interested in making significant contributions to studying the stability of systems of fractional differential equations of linear type.

For the nonlinear case, stability analysis is much more complex; however, some authors have studied the system of nonlinear differential equations of fractional order using various approaches, and some stability analysis methods for the ordinary case extended to fractional order. In 2007, Tarasov mentioned that the concept of stability concerning fractional variations is broader than that in the classical sense of Lyapunov or asymptotic stability. He states that fractional stability includes the concept of classical stability as a specific case $\alpha=1$ and that systems can be unstable concerning the first variation of states and stable concerning fractional variation. Therefore, fractional derivatives extend the possibility of investigating the properties of systems of differential equations [16].

In 2009, [17] analyzed the stability of systems of nonlinear ordinary differential equations of fractional type using the derivative operator in the Caputo or Riemann– Liouville sense for order $0<\alpha<1$. They succeeded in performing an extension of classical order stability concepts to systems of fractional order differential equations, establishing that the drop in the generalized energy of a dynamical system does not have to be exponential for the system to be stable; this decrease in energy can be of any type, including a power law decrease.

To extend the application of fractional calculus to nonlinear systems, they proposed stability in the sense of the direct fractional method of Mittag-Leffler and Lyapunov [18]. In 2010, the same authors generalized the concepts of stability in the fractional-type Mittag–Leffler and Lyapunov sense, providing fundamental contributions to asymptotic-type stability for the same order $0<\alpha<1$ [19].

The remarkable progress in the theoretical results of fractional differential equations focused on the study of stability is due to the applicability of modeling a problem through fractional operators. Among the fundamental problems modeled through fractional operators and that have obtained better responses in their modeling are the biological models of the prey–predator type, also called Lotka– Volterra type models [20-27]. These are of great importance for analyzing the interaction between two or more species on the dynamics between predators and their respective prey.

This article presents sections structured as follows: the next section presents the fundamental preliminaries of fractional operators and the properties of differential equations of fractional order; then, the fundamental concepts of stability of systems of ordinary differential equations, both linear and non-linear, are presented; and the next section presents a description of the

Lotka–Volterra type models studied and an analysis and discussion of the stability analysis of the studied model conducted.

2. Preliminaries

In this section, we will initially present the fundamental preliminaries on fractional analysis and then present the relevant concepts for the study of ordinary differential equations of fractional order using derivative operators in the sense of Riemann–Liouville, Caputo, and Hadamard. The core references in this section are taken from $[\,28-35]$.

In this article, $\mathbb{Z}_+$ denotes the set of positive integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ represents the set of complex numbers, $\mathbb{R}e(\alpha)$ represents the real part of the complex number α and $\Gamma(\cdot)$ represents the Gamma function.

Definition 1: The fractional integral in the Riemann–Liouville sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) > 0$) of the function $x(t)$ denoted by $\mathcal{D}_{t_0,t}^{-\alpha}[x(t)]$ or $\mathcal{J}_{t_0,t}^{\alpha}[x(t)]$, is given by:

$$\mathcal{J}_{t_0,t}^{\alpha}[x(t)] = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{x(s)}{(t-s)^{1-\alpha}}\,ds, t > t_0 \quad (1)$$

Definition 2: The fractional derivative in the Riemann–Liouville sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) \geq 0$) of the function $x(t)$ denoted by $\mathcal{D}_{t_0,t}^{\alpha}[x(t)]$, is given by

$$\begin{aligned}\mathcal{D}_{t_0,t}^{\alpha}[x(t)] &= \frac{d^n}{dt^n}\left(\mathcal{J}_{t_0,t}^{n-\alpha}[x(t)]\right)\\ &= = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{t_0}^{t}\frac{x(s)}{(t-s)^{\alpha+1-n}}\,ds, \quad t > t_0\,,\end{aligned} \quad (2)$$

where $n-1 < \alpha < n$, with $n \in \mathbb{Z}_+$.

Caputo's fractional derivative allows the consideration of initial conditions of natural order to mathematical models represented by fractional differential equations. These fractional derivative operators are defined as follows:

Definition 3: The fractional derivative in the Caputo sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) \geq 0$) of the function $x(t)$ denoted by ${}_{c}\mathfrak{D}_{t_0,t}^{\alpha}[x(t)]$, is given by

$$\begin{aligned}{}_{c}\mathfrak{D}_{t_0,t}^{\alpha}[x(t)] &= \mathcal{J}_{t_0,t}^{n-\alpha}\left[x^{(n)}(t)\right]\\ &= \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{x^{(n)}(s)}{(x-s)^{\alpha+1-n}}\,ds, t > t_0,\end{aligned} \quad (3)$$

where $x^{(n)}(s)$ represents the nth derivative of the function $x(s)$ and $n-1 < \alpha < n$, with $n \in \mathbb{Z}_+$.

Just as in the solutions of ordinary differential equations of integer order, the exponential function plays a fundamental role. In addition, in the solutions of differential equations of fractional order, an analogous function called the Mittag–Leffler function is used.

Definition 4: The Mittag–Leffler function of a parameter α is given by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + 1)}, \quad (4)$$

where $z \in \mathbb{C}$, $\mathbb{R}e(\alpha) > 0$. The Mittag–Leffler function of two parameters α and β is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (5)$$

where $z, \beta \in \mathbb{C}$, $\mathbb{R}e(\alpha) > 0$.

When $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the one-parameter Mittag–Leffler function, that is, $E_{\alpha,1}(z) = E_{\alpha}(z)$. Moreover, when $\alpha = \beta = 1$, the function $E_{1,1}(z)$ is equivalent to the function e^z.

Proposition 1: If $0 < \alpha < 2$ and $\beta \in \mathbb{C}$ arbitrary, then for a $n \in \mathbb{Z}$ with $n \geq 1$ we have the following expansions:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}z^{\frac{1-\beta}{\alpha}}e^{z^{\frac{1}{\alpha}}} - \sum_{k=1}^{n}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k} + O\left(\frac{1}{|z|^{n+1}}\right),$$

with $|z| \to \infty$, $|\arg(z)| \leq \alpha\frac{\pi}{2}$,

and

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{n}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k} + O\left(\frac{1}{|z|^{n+1}}\right),$$

with $|z| \to \infty$, $|\arg(z)| > \alpha\frac{\pi}{2}$.

The following results can be considered generalizations of the exponential function.

Definition 5: The function $e_{\alpha}^{\lambda z} = z^{\alpha-1}E_{\alpha,\alpha}(\lambda z^{\alpha})$ is called $\alpha-$exponential function, where$z \in \mathbb{C}\backslash\{0\}, \lambda \in \mathbb{C}$ and $\mathbb{R}e(\alpha) > 0$.

According to proposition 1 and definition 5, it follows that $E_{1,1}(z) = E_1(z)$ and indeed $e_1^{\lambda z} = e^{\lambda z}$.

From definition 5 of $\alpha-$exponential function and proposition 1, we have the following result:

Proposition 2: If $0 < \alpha < 2$ and $z \in \mathbb{C}$ then one has the following asymptotic equivalences for the $\alpha-$exponential function:

For $|\arg(z)| \leq \alpha\frac{\pi}{2}$, you must $e_{\alpha}^{\lambda z} \sim \frac{\lambda^{\frac{1-\alpha}{\alpha}}}{\alpha}e^{\lambda^{\frac{1}{\alpha}}}z$ when $|z| \to \infty$.

For $|\arg(z)| > \alpha\frac{\pi}{2}$, you must $e_{\alpha}^{\lambda z} \sim -\frac{1}{\lambda^2\Gamma(-\alpha)}\frac{1}{z^{\alpha+1}}$ when $|z| \to \infty$.

Proposition 3: If $0 < \alpha < 2$, $\beta > 0$, $|z| > 1$, μ satisfies $\alpha\frac{\pi}{2} < \mu < min\{\alpha\pi, \pi\}$ and $C > 0$ is a very constant, then $|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}$, where $\mu \leq |\arg(z)| \leq \pi$.

Proposition 4: If $A \in \mathbb{C}^{n\times n}$, $0 < \alpha < 2$, β is an arbitrary real number, μ in such that $\alpha\frac{\pi}{2} < \mu < min\{\alpha\pi, \pi\}$ and $C > 0$ is very constant, then $\| E_{\alpha,\beta}(A) \| \leq \frac{C}{1+\|A\|}$, where $\mu \leq |\arg[spec(A)]| \leq \pi$.

The following are the fundamental definitions of ordinary differential equations of fractional order.

Definition 6: An ordinary differential equation of fractional order α is defined as follows:

$$\mathbb{D}^{\alpha}_{t_0,t}[x] = f\left(t, x, \mathbb{D}^{\alpha_1}_{t_0,t}[x], \mathbb{D}^{\alpha_2}_{t_0,t}[x], \ldots, \mathbb{D}^{\alpha_{m-1}}_{t_0,t}[x]\right) \quad (6)$$

where $x(t)$ is a real domain complex unknown function, $f(t, x, x_1, x_2, x_3, \ldots, x_{n-1})$ is a known function and $\mathbb{D}^{\alpha_k}_{t_0,t}$ for $k = 1,2,3, \ldots, m-1$ are fractional differential operators such that $0 < \mathbb{R}e(\alpha_1) < Re(\alpha_2) < Re(\alpha_3) < \cdots < Re(\alpha_{m-1}) < Re(\alpha)$ for $m \in \mathbb{Z}_+, m \geq 2$.

Definition 7: An ordinary differential equation of fractional order α of linear type is defined as follows:

$$\mathbb{D}^{\alpha}_{t_0,t}[x] + a_0(t)x + \sum_{k=1}^{m-1} a_k(t)\mathbb{D}^{\alpha_k}_{t_0,t}[x] = f(t) \quad (7)$$

where $x(t)$ is a real domain complex unknown function, $a_k(t)$ for $k = 0,1,2,3, \ldots, m-1$ and $f(t)$ are known functions, $\mathbb{D}^{\alpha}_{t_0,t}$ for $k = 1,2,3, \ldots, m-1$ are fractional differential operators such that $0 < \mathbb{R}e(\alpha_1) < Re(\alpha_2) < Re(\alpha_3) < \cdots < Re(\alpha_{m-1}) < Re(\alpha)$ for $m \in \mathbb{Z}_+, m \geq 2$.

If the functions $a_k(t)$ are constant for all $k = 0,1,2,3, \ldots, m-1$, equation (7) is said to have constant coefficients. On the contrary, if at least one of them is variable, the equation is said to have variable coefficients. Now, if $f(t) = 0$ the linear fractional differential equation of the ordinary type is said to be homogeneous.

The Cauchy-type problem for the Riemann–Liouville fractional derivative presented in equation (1) is given by

$$\begin{cases} \mathcal{D}^{\alpha}_{t_0,t}[x(t)] & = f\left(t, x, \mathcal{D}^{\alpha_1}_{t_0,t}[x], \mathcal{D}^{\alpha_2}_{t_0,t}[x], \ldots, \mathcal{D}^{\alpha_{m-1}}_{t_0,t}[x]\right) \\ \mathcal{D}^{\alpha-k}_{t_0,t}[x(t)]|_{t=t_0} & = b_k, \quad b_k \in \mathbb{C}, \quad k = 1,2,3, \ldots, m. \end{cases} \quad (8)$$

where $m = \begin{cases} [\mathbb{R}e(\alpha)] + 1 & if \quad \alpha \notin \mathbb{Z}_+ \\ \alpha & if \quad \alpha \in \mathbb{Z}_+ \end{cases}$.

The Cauchy problem for the Caputo fractional derivative presented in equation (2) has the following structure:

$$\begin{cases} {}_c\mathfrak{D}^{\alpha}_{a^+,t}[x] & = f\left(t, x, {}_c\mathfrak{D}^{\alpha_1}_{a^+,t}[x], {}_c\mathfrak{D}^{\alpha_2}_{a^+,t}[x], \ldots, {}_c\mathfrak{D}^{\alpha_{m-1}}_{a^+,t}[x]\right) \\ x^{(k)}(a) & = b_k, \quad b_k \in \mathbb{C}, \quad k = 1,2,3, \ldots, m. \end{cases} \quad (9)$$

where $0 < \mathbb{R}e(\alpha_1) < Re(\alpha_2) < Re(\alpha_3) < \cdots < Re(\alpha_{m-1}) < Re(\alpha)$ for $m \in \mathbb{Z}_+, m \geq 2$.

In this case, the initial conditions are given in terms of ordinary derivatives, which facilitate the physical interpretation.

The ordinary differential equation of fractional order presented in Equation (6) can be expressed as follows:

$$\mathbb{D}^{\tilde{\alpha}}_{t_0,t}[x(t)] = f\left(t, x(t)\right), \quad (10)$$

where $f: [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\tilde{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$ with $m - 1 < \alpha_i < m$ for $m \in \mathbb{Z}_+$ and $i = 1,2, \ldots, n$, $x_k = [x_{k_1}, x_{k_2}, \ldots, x_{k_n}]^T \in \mathbb{R}^n$ for $k = 0,1,2, \ldots, m-1$ represent suitable initial conditions where$x(t) = [x_1(t), x_2(t) \ldots, x_n(t)]^T \in \mathbb{R}^n$. The derivative operator denoted by $\mathbb{D}^{\tilde{\alpha}}_{t_0,t}$ represents the derivative in the Caputo or Riemann–Liouville sense.

If $\alpha_1, \alpha_2, \ldots, \alpha_n = \alpha$ then Equation (10) can be written as

$$\mathbb{D}^{\alpha}_{t_0,t}[x(t)] = f\left(t, x(t)\right), \quad (11)$$

and is called a system of fractional differential equations of the same order.

Definition 8: A constant vector x_* is said to be the equilibrium point of the system of fractional differential equations (10), if and only if $f(t, x_*) = \mathbb{D}^{\tilde{\alpha}}_{t_0,t}[x(t)]|_{x(t)=x_*}$ for all $t > t_0$.

Without loss of generality, we can consider the equilibrium point at the origin, that is, $x_* = 0$ and establish the following definition.

Definition 9: The solution $x_* = 0$ of the system of fractional order differential equations (11), is given by

1. Stable, if for any initial condition $x_k = [x_{k_1}, x_{k_2}, \ldots, x_{k_n}]^T \in \mathbb{R}^n$ with $k = 0,1,2, \ldots, m-1$, there exists $\epsilon > 0$ such that any solution $x(t)$ of equation (11) satisfies that$||x(t)|| < \epsilon$ for all $t > t_0$.
2. Asymptotically stable if it is stable and is satisfied that $||x(t)|| \to 0$ when $t \to +\infty$.

3. Stability of Fractional Ordinary Differential Equations (FDEs)

In this section, we consider the main results that have been presented on the stability of systems of ordinary differential equations of fractional order of linear and nonlinear types. The core references in this section are taken from $[28 - 29,\ 35 - 51]$.

For this, we consider the linear system of FDEs,

$$\mathbb{D}^{\tilde{\alpha}}_{t_0,t}[x(t)] = Ax(t), \quad (12)$$

where $x(t) = [x_1(t), x_2(t) \ldots, x_n(t)]^T \in \mathbb{R}^n$, A is a matrix such that $A \in \mathbb{R}^{n\times n}$, $\tilde{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$, $\mathbb{D}^{\tilde{\alpha}}_{t_0,t}[x(t)] = \left[\mathbb{D}^{\alpha_1}_{t_0,t}[x_1(t)], \mathbb{D}^{\alpha_2}_{t_0,t}[x_2(t)], \ldots, \mathbb{D}^{\alpha_n}_{t_0,t}[x_n(t)]\right]^T$.

The derivative operator by notation $\mathbb{D}^{\alpha_i}_{t_0,t}$ represents the derivative in the sense of Caputo or Riemman–Liouville of order α_i, where $0 < i \leq 2$ for $i = 1,2, \ldots, n$.

If $\alpha_1, \alpha_2, \ldots, \alpha_n = \alpha$ then the Equation (12) can be written as

$$\mathbb{D}^{\alpha}_{t_0,t}[x(t)] = Ax(t). \quad (13)$$

The first stability results for linear FDE systems were presented by in [15] using tools from an algebraic approach and using asymptotic results applied to control theory. Said result on stability was established for the system (13) for the order $0 < \alpha \leq 1$, which is presented below.

Theorem 1: The autonomous system (13) with the Caputo derivative and initial value $x_0 = x(0)$, where $0 < \alpha \leq 1$, is:

1. Asymptotically stable if and only if $|\arg[spec(A)]| > \alpha\frac{\pi}{2}$. In this case, the state components decay toward 0 as $\frac{1}{t^{\alpha}}$.
2. Stable if and only if either it is asymptotically stable or those critical eigenvalues which satisfy

$|\arg[spec(A)]| = \alpha\frac{\pi}{2}$ have geometric multiplicity one where $spec(A)$ denotes the eigenvalues of the matrix A corresponding to system (13).

From the classical theory of stability for systems of ordinary differential equations of the linear type of integer order, we know that we can establish the stability of a linear system at its equilibrium point by studying the eigenvalues of the matrix associated A to the system. This algebraic study establishes that if the roots of the characteristic polynomial associated with the matrix A have a negative real part, then the linear system is asymptotically stable.

The result presented in [15] in Theorem 1 for the case where $0 < \alpha < 1$, shows that the roots of the characteristic polynomial associated with the matrix A of the system (13) lie outside the closed angular sector $|\arg[spec(A)]| \leq \alpha\frac{\pi}{2}$, as shown in Fig. 1.

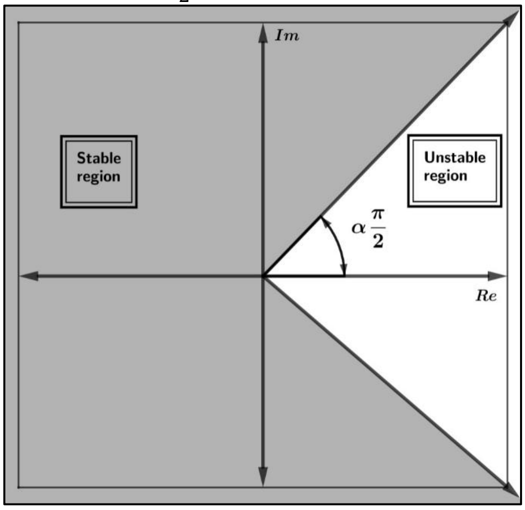

Fig. 1 Stability region for $0 < \alpha < 1$ (Developed by the authors)

For the case where $\alpha = 1$ we have the stability region presented in the classical sense of linear ordinary differential equations of natural order, as shown in Fig. 2.

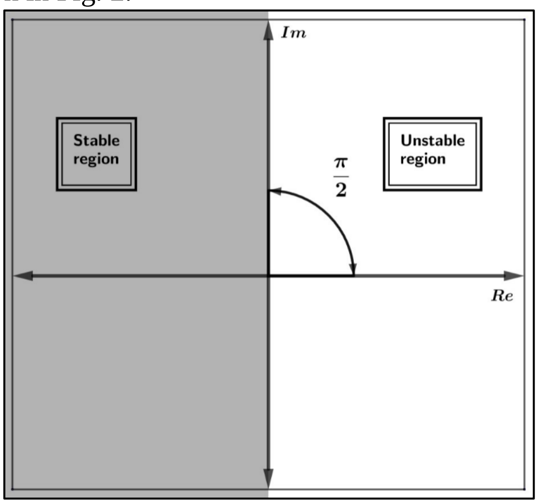

Fig. 2 Stability region for $\alpha = 1$ (Developed by the authors)

Note that the asymptotic stability of the system (13) presented in Theorem 1 is also called $t^{-\alpha}-$stability because the state components exhibit anomalous decay.

On the other hand, Qian et al. proved Theorem 1. without using control theory entities and for the non-asymptotic stability of the system (13) and presented the following theorem.

Theorem 2: If all the eigenvalues of the matrix A satisfy $|\arg[spec(A)]| \geq \alpha\frac{\pi}{2}$ and the critical eigenvalues satisfying $|\arg[spec(A)]| = \alpha\frac{\pi}{2}$ have the same algebraic and geometric multiplicities, then the zero solution of system (13) is stable but not asymptotically stable.

Furthermore, for system (13) [42] studied the case when there are null eigenvalues in the matrix A of the linear system using the Riemann–Liouville derivative operator with asymptotic expansions of the Mittag–Leffler function for the order $0 < \alpha < 1$, of the following way.

Theorem 3: The system (13) with the Riemann–Liouville derivative and initial value $x_0 = \mathcal{D}_{t_0,t}^{\alpha-1}[x(t)]|_{t=t_0}$, where $0 < \alpha < 1$ and $t_0 = 0$, is

1. Asymptotically stable if and only if all non- zero eigenvalues of the matrix A satisfy $|\arg[spec(A)]| > \alpha\frac{\pi}{2}$, or the A has $k-$multiple zero eigenvalues corresponding to Jordan block $diag(J_1, J_2, \ldots, J_i)$, where J_l is a Jordan canonical form with order n_l, for $\sum_{l=1}^{i} n_l = k$, and $n_l\alpha < 1$, for $1 \leq l \leq i$.
2. Stable if and only if either it is asymptotically stable, those critical eigenvalues which satisfy $|\arg[spec(A)]| = \alpha\frac{\pi}{2}$ have the same algebraic and geometric multiplicities, or the A has $k-$multiple zero eigenvalues corresponding to a Jordan block $diag(J_1, J_2, \ldots, J_i)$, where J_l is a Jordan canonical form with order n_l, for $\sum_{l=1}^{i} n_l = k$, and $n_l\alpha < 1$, for $1 \leq l \leq i$.

In the above theorem, the state components decay toward 0 as $t^{-\alpha-1}$ if all eigenvalues of the system matrix A satisfy $|\arg[spec(A)]| > \alpha\frac{\pi}{2}$. If all the nonzero eigenvalues of the matrix A satisfy $|\arg[spec(A)]| \geq \alpha\frac{\pi}{2}$ and the critical eigenvalues that satisfy $|\arg[spec(A)]| = \alpha\frac{\pi}{2}$ have the same algebraic and geometric multiplicities, and the zero eigenvalue of matrix A has the same algebraic and geometric multiplicities, then the null solution of system (13) is stable from the representation of the solution.

For the multiple-order linear fractional system of differential equations, as expressed in Equation (13), Deng et al. presented the following output [43].

Theorem 4: Suppose that α_i's are rational numbers between 0 and 1, for $i = 1,2,\ldots,n$. Let M be the lowest common multiple (LCM) of the denominators u_i of α_i's, where $\alpha_i = \frac{v_i}{u_i}$,$(u_i, v_i) = 1$, u_i, $v_i \in \mathbb{Z}_+, i = 1,2,\ldots,n$, and set $\gamma = \frac{1}{M}$. Then, the zero solution of system (1) with the Caputo derivative and initial value

$x_0 = x(0)$ is:

1. Asymptotically stable if and only if any zero solution of the polynomial $\det[diag(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \dots, \lambda^{M\alpha_n}) - A]$satisfies $|\arg(\lambda)| > \gamma\frac{\pi}{2}$, the components of the state variable $[x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n$ decay toward 0 like $t^{-\alpha_1}, t^{-\alpha_2}, \dots, t^{-\alpha_n}$, respectively.
2. Stable if and only if either it is asymptotically stable or the critical zero solutions λ of the above polynomial satisfy $|\arg(\lambda)| > \gamma\frac{\pi}{2}$ have geometric multiplicity one.

Note that if $\alpha_1, \alpha_2, \dots, \alpha_n = \alpha$ are rational numbers for $0 < \alpha_i < 1$ with $i = 1,2, \dots, n$, then Theorem 4 coincides with Theorem 1 presented by [15]; that is, the previous theorem is an extension of Theorem 1 with respect to rational orders.

[44], like [45], also analyzed the stability of the system (13) for the case $0 < \alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_n \leq 1$ using Mittag–Leffler functions and their integer order derivatives to obtain analytic solutions of the initial value problem (13) and then establish the condition of sufficient stability using the final value theorem.

All the above conclusions refer to the case of commensurable fractional order. Moreover, in [45], they also study the case of incommensurable fractional order. If $\alpha_1, \alpha_2, \dots, \alpha_n$ are irrational numbers between 0 and 1 in the system of differential equations (13), we obtain the following result.

Theorem 5: If all the roots of the characteristic equation $\det[diag(s^{\alpha_1}, s^{\alpha_2}, \dots, s^{\alpha_n}) - A] = 0$ have negative real parts, then the zero solution of system (13) is asymptotically stable, where α_i is real and lies in (0,1).

From the previous theorems and Proposition 1, we obtain the following result.

Theorem 6: The autonomous same order system (13) with initial value $x_0 = x(0)$ and Riemann–Liouville derivative is asymptotically stable if and only if $|\arg[spec(A)]| > \alpha\frac{\pi}{2}$, where $n = 2$ and $0 < \alpha \leq 1$.

The proof of the previous theorem can be consulted in [46], which uses from [29] the unique solution of the system (13) expressed as a generalization of the matrix α −exponential.

The initial results on the stability of linear fractional order differential equation systems, based on the results proposed in [15], have been generalized to the order $0 < \alpha < 2$ by some researchers but from different approaches. In $[26 - 28]$ they study the stability case for the linear system (13) for the order $1 < \alpha < 2$ using control theory tools such as linear matrix inequality (LMI) methods and establish the region of stability as a generalization of the results proposed by [15].

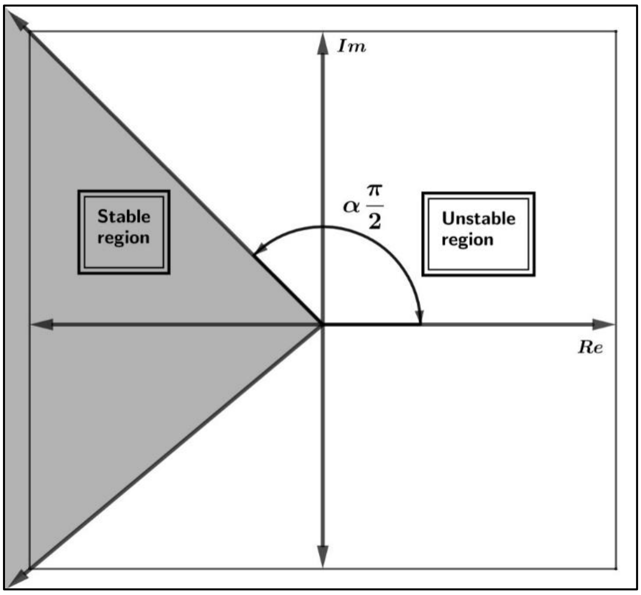

Fig. 3 Stability region for $1 < \alpha < 2$ (Developed by the authors)

Indeed, for the case where $\alpha = 1$, we have the region of stability that we know in the classical sense of the derivative of integer order, as shown in Fig. 2.

In [47], the stability result for the order $1 < \alpha < 2$ is also generalized by performing an extension of Theorem 1, to obtain the following result.

Theorem 7: The autonomous same order system (13) with Riemann–Liouville derivative and initial values $x_k = \mathcal{D}_{0,t}^{\alpha-k-1}[x(t)]|_{t=0}$, for $k = 0,1$, is asymptotically stable if and only if $|\arg[spec(A)]| > \alpha\frac{\pi}{2}$, where $n = 2$ and $1 < \alpha < 2$.

The proof can be found in [29] or [47]. See [48] and [49] which studied the stability of linear FDE systems using various techniques for stability analysis.

In [50], the authors discuss the stability of the following linear FDE system with the Riemann–Liouville derivative operator is expressed as

$$\begin{cases} \mathcal{D}_{0,t}^{\alpha}[x(t)] & = Ax(t) + B(t)x(t) \\ \mathcal{D}_{0,t}^{\alpha-1}[x(t)]|_{t=0} & = x_0, \end{cases} \quad (14)$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^n \times \mathbb{R}^n$ is a matrix and $B(t): [0, \infty] \to \mathbb{R}^n \times \mathbb{R}^n$ is a continuous matrix. For system (3), sufficient conditions are given to establish stability and asymptotic stability using the Mittag–Leffler function, the generalized Gronwall inequality and the comparison principle for orders $0 < \alpha < 1$ and $1 < \alpha < 2$.

In [51], the authors conduct a detailed analysis of the stability of systems of linear FDEs using the derivative operator in Caputo's sense. To do so, they initially present a new proof of Theorem 1, proposed by [15], to later study the case of systems of linear FDEs of multiple orders.

3.1. Stability of the Nonlinear FDE

We consider the main results that have been presented for the stability of systems of ordinary differential equations of nonlinear fractional order. As we mentioned at the beginning, there is still no established theory of stability for FDEs similar to that of integer-order ordinary differential equations. However, fundamental stability results have been presented for systems of nonlinear differential equations.

In [52, 53, and 54], the authors began to study the stability of nonlinear FDE systems using Lyapunov functions for various generalizations of fractional derivative operators. In [55, 56], they managed to extend the exponential stability that we know in the ODEs of integer order [57, 58] to the so-called Mittag–Leffler stability, and with it, they introduced the direct Lyapunov method of fractional order using the fractional derivative operator in the sense of Riemann–Liouville and Caputo.

Therefore, the authors initially presented the following definitions [55, 56].

Definition 10: A solution $x(t)$ of the system (11) of nonlinear fractional differential equations is said to be Mittag–Leffler stable if it satisfies the following:

$$\|x(t)\| \leq \{m[x(t_0)]E_\alpha(-\lambda(t-t_0)^\alpha)\}^b \qquad (15)$$

where: E_α is the one-parameter Mittag–Leffler function, t_0 is the initial time, $\alpha \in (0,1)$, $\lambda \geq 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally lipschitzian on $x \in B \in \mathbb{R}^n$ with Lipschitz constant m_0.

The same authors generalized Definition 10 as follows.

Definition 11: A solution $x(t)$ of the system (11) of nonlinear fractional differential equations is said to be generalized Mittag–Leffler stable if it satisfies the following:

$$\|x(t)\| \leq \{m[x(t_0)](t-t_0)^{-\gamma}E_{\alpha,1-\gamma}(-\lambda(t-t_0)^\alpha)\}^b \qquad (16)$$

where $E_{\alpha,1-\gamma}$ is the two-parameter Mittag–Leffler function, t_0 is the initial time, $-\alpha < \gamma \leq 1-\alpha$, $\alpha \in (0,1)$, $\lambda \geq 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally lipschitzian on $x \in B \subset \mathbb{R}^n$ with Lipschitz constant m_0.

The following results follow from the above definitions.

Remark 1: It follows from the properties of the Mittag–Leffler function and the completely monotonic function that $\mathcal{D}_{0,t}^{\gamma}[E_\alpha(-\lambda t^\alpha)] = \mathbb{D}_{0,t}^{\gamma}[E_\alpha(-\lambda t^\alpha)] = {}_{\mathrm{c}}\mathfrak{D}_{a^+,t}^{\gamma}[E_\alpha(-\lambda t^\alpha)] = t^{-\gamma}E_{\alpha,1-\gamma}(-\lambda t^\alpha)$ is a completely monotonic function for $\alpha \in (0,1)$, $\lambda \geq 0$ and $\gamma \in [0, 1-\alpha]$.

Remark 2: Mittag–Leffler stability and generalized Mittag–Leffler stability imply asymptotic stability.

Remark 3: Let $\lambda = 0$, it follows from equation (14) that $\|x(t)\| \leq \left[\frac{m(x(t_0))}{\Gamma(1-\gamma)}\right]^b (t-t_0)^{-\gamma b}$, which implies that power law stability is a special case of Mittag–Leffler stability.

Next, the extension of the direct Lyapunov method to the case of fractional order systems is presented, leading to Mittag–Leffler stability [56].

Theorem 8: Let $x = 0$ be an equilibrium point of the system (10) and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $L(t, x(t)): [0,\infty) \times D \longrightarrow \mathbb{R}$ be a continuously differentiable and locally Lipschitzian function with respect to x, such that $\beta_1\|x\|^a \leq L(t,x(t)) \leq \beta_2\|x\|^{ab}$, ${}_{\mathrm{c}}\mathfrak{D}_{0,t}^{\alpha}L(t,x(t)) \leq -\beta_3\|x\|^{ab}$, where $t \geq 0$, $x \in \mathbb{D}$, $\alpha \in (0,1)$ and $a, b, \beta_1, \beta_2, \beta_3$ are arbitrary positive constants.

If the function $L(t, x(t))$ satisfies the previous inequalities, then it is said to be a fractional Lyapunov function, and in effect, the equilibrium point $\mathrm{x} = 0$ is said to be Mittag–Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, $x_* = 0$ is globally Mittag–Leffler stable.

Considering Remark 2, the authors of [56, 57] obtained an asymptotic stability result directly by weakening conditions in Theorem 8.

Theorem 9: Let $x = 0$ be an equilibrium point of the system (11) and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $L(t, x(t)): [0,\infty) \times D \longrightarrow \mathbb{R}$ be a continuously differentiable and locally Lipschitzian function with respect to x, such that $\beta_1\|x\|^a \leq L(t,x(t)) \leq \beta_2 {}_{\mathrm{c}}\mathfrak{D}_{0,t}^{-\eta}\|x\|^{ab}$, ${}_{\mathrm{c}}\mathfrak{D}_{0,t}^{\alpha}L(t,x(t)) \leq -\beta_3\|x\|^{ab}$, where $t \geq 0$, $x \in \mathbb{D}$, $\alpha \in (0,1)$, $\eta > 0, \eta \neq \alpha, |\alpha - \eta| < 1$ and $a, b, \beta_1, \beta_2, \beta_3$ are positive arbitrary constants. If the function $L(t, x(t))$ satisfies the previous inequalities, then it is said to be a fractional generalized Lyapunov function, and in effect, the equilibrium point $\mathrm{x}_* = 0$ is asymptotically stable.

The asymptotic stability of the system (11) can also be obtained through a particular type of function, as presented below.

Definition 12: A continuous function $\alpha: [0, a) \longrightarrow [0, \infty)$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

Theorem 10: Let $x = 0$ be an equilibrium point of the system (11). If they exist, a Lyapunov function $L(t, x(t))$ and class functions $\mathcal{K}$, α_1, α_2 and α_3 that satisfy $\alpha_1\|x\| \leq L(t, x(t)) \leq \alpha_2\|x\|$, ${}_{\mathrm{c}}\mathfrak{D}_{0,t}^{\beta}L(t,x(t)) \leq -\alpha_3\|x\|$, where $\beta \in (0,1)$. The equilibrium point of system (11) is asymptotically stable.

The respective proofs to the above theorems can be found in [56, 57].

The following result allows us to establish the stability of one type of system of nonlinear ordinary differential equations.

Theorem 11: Let $\alpha \in (0,2)$ and the initial value problem

$$ {}_{\mathrm{c}}\mathfrak{D}_{0,t}^{\alpha}[x(t)] = f(t, x(t)) \qquad (17)$$

$$x^k(0) = x_k, k = 0,1,$$

a system of nonlinear fractional differential equations. If the following two conditions hold true:

1. $|\arg(spec(A))| > \alpha\frac{\pi}{2}$,

2. $\frac{\| f(x(t)) \|}{\| x(t) \|} \to 0$ when $\| x(t) \| \to 0$,

Then the system ${}_{c}\mathfrak{D}^{\alpha}_{0,t}[x(t)] = Ax(t) + f(x(t))$, es locally asymptotically stable at $x = 0$, where $\alpha \in (0,2)$, $A \in \mathbb{R}^{n\times n}$ and $f: \mathbb{D} \longrightarrow \mathbb{R}^n$.

Proof: The solution of the fractional differential equation ${}_{c}\mathfrak{D}^{\alpha}_{0,t}[x(t)] = Ax(t) + f(x(t))$, is given by $x(t) = x_0 E_{\alpha,1}(At^{\alpha}) + x_1 t E_{\alpha,2}(At^{\alpha}) + \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^{\alpha}) f(x(\tau)) d\tau$.

By the condition 2., the exist $C_2 > 0$ and $\delta > 0$, such that $\| f(x(t)) \| < \frac{(\alpha-1)\|A\|}{2C_2} \| x(t) \|$ as $\| x(t) \| < \delta$.

From Propositions 4 and 1, we have $\| x(t) \| \leq \frac{C_0 \|x_0\|}{1+\|At^{\alpha}\|} + \frac{C_1 \|x_1 t\|}{1+\|At^{\alpha}\|} + \int_0^t \frac{\|(t-\tau)^{\alpha-1}\| C_2 (\alpha-1)\|A\|}{2C_2(1+\|A(t-\tau)^{\alpha}\|)} \| x(\tau) \| d\tau$, where $C_0 > 0$, $C_1 > 0$.

By the Gronwall–Bellman lemma, we have

$$+\frac{C_1 \| x_1 \| t}{1+\| At^{\alpha} \|} + \int_0^t \left(\frac{C_0 \| x_0 \|}{(1+\| A \| t^{\alpha})^{\frac{1}{2}}} + \frac{C_1 \| x_1 \| t}{1+\| At^{\alpha} \|} \right) \frac{(\alpha-1) \| A \| (t-\tau)^{\alpha-1}}{2(1+\| A \| (t-\tau)^{\alpha})} \times \exp\left(\int_{\tau}^{t} \frac{(\alpha-1) \| A \| (t-s)^{\alpha-1}}{2(1+\| A \| (t-s)^{\alpha})} ds \right) d\tau$$

$$\leq \frac{C_0 \| x_0 \|}{(1+\| A \| t^{\alpha})^{\frac{1}{2}}} \| x(t) \|$$

$$\leq \frac{C_0 \| x_0 \|}{(1+\| A \| t^{\alpha})^{\frac{1}{2}}} + \frac{C_1 \| x_1 \| t}{1+\| A \| t^{\alpha-1}} + \frac{(\alpha-1) C_0 \| x_0 \| \Gamma\left(1-\frac{1}{2\alpha}\right) \Gamma\left(\frac{1}{2(\alpha-1)}\right)}{2 \| A \|^{\frac{1}{2\alpha}} \Gamma\left(\frac{1}{2}\right) t^{\frac{1}{2}}}$$

$$\frac{(\alpha-1) C_1 \| x_1 \| \Gamma(2-\alpha) \Gamma\left(\frac{1}{2(\alpha-1)}\right)}{2 \| A \|^{\frac{1}{2(\alpha-1)}} \Gamma\left(\frac{3}{2}-\frac{1}{2\alpha}\right) t^{\frac{1}{2(\alpha-1)}}} \longrightarrow 0 \ as \ t \longrightarrow \infty.$$

Thus, the zero solutions of the system FDE is locally asymptotically stable.

4. Stability Analysis of the Fractional Lotka–Volterra Model

In the 1920s, mathematicians Alfred Lotka and Vito Volterra simultaneously but independently proposed a nonlinear differential equation model to describe the population dynamics of two interacting species, a predator and its prey. They hoped to explain the increase in predators and, therefore, the decrease in prey in various ecological environments [20].

The model also called the prey predator system is given by

$$\begin{cases} \dot{P} & = & \varepsilon P - \delta P D \\ \dot{D} & = & -\gamma D + \beta D P, \end{cases} \qquad (18)$$

where $P(t)$ and $D(t)$ represent the prey and predator populations at time t respectively; parameters $\varepsilon, \delta, \gamma$ and β are positive constants. ε represents the natural growth rate of the prey in the absence of a predator, δ represents the effect of the predator on the prey population, β represents the effect of the prey on the predator population and γ represents the natural mortality rate of the predator in the absence of a prey. It is noteworthy that because we are dealing with population dynamics, we only consider $P, D > 0$.

Fig. 4 shows the graphs of $P(t)$ and $D(t)$ where the interaction between prey and predators is observed, and Fig. 5 shows the trajectories near the equilibrium point D.

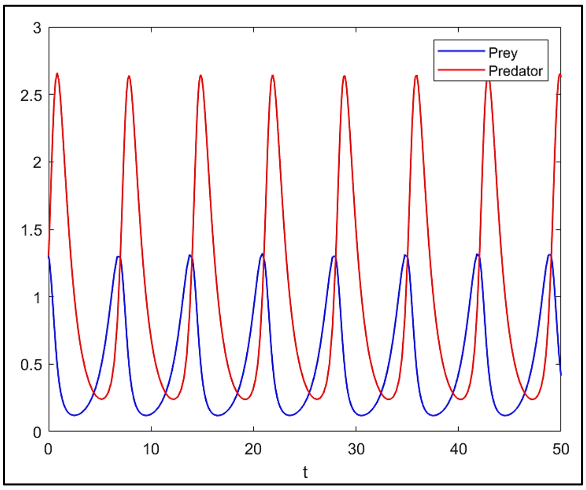

Fig. 4 Graphs of $P(t)$ and $Q(t)$ (Developed by the authors)

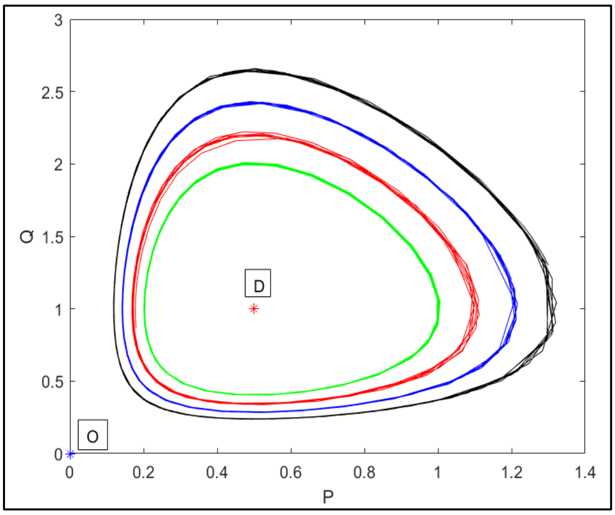

Fig. 5 Trajectories for system (18) (Developed by the authors)

It is noteworthy that system (18) is stable but not asymptotically stable, as shown in Fig. 5. The trajectories orbit around the equilibrium point D.

The prey–predator model where two species interact (18) expressed in its fractional version using the derivative operator in the Caputo sense, of the same order, is given by:

$$\begin{cases} {}_{c}\mathfrak{D}_{0,t}^{\alpha}[P(t)] &= \varepsilon P(t) - \delta P(t)D(t) \\ {}_{c}\mathfrak{D}_{0,t}^{\alpha}[D(t)] &= -\gamma D(t) + \beta D(t)P(t), \end{cases} \quad (19)$$

where $P(t)$ and $D(t)$ represent the prey and predator populations at time t respectively; parameters $\varepsilon, \delta, \gamma$ and β are positive constants.

We will not go too deeply into the stability analysis of system (19). Initially, we locate the equilibrium points of system (19), which occur when $\varepsilon P(t) - \delta P(t)D(t) = 0$ and $-\gamma D(t) + \beta D(t)P(t) = 0$, which are located at the origin $O = (0,0)$ and at $D = (\gamma/\beta, \varepsilon/\delta)$.

Let us analyze the stability at equilibrium point D, the linearized flow at this point is given by $\mathcal{J}\left(\frac{\gamma}{\beta}, \frac{\varepsilon}{\delta}\right) = \begin{bmatrix} 0 & \frac{-\delta\gamma}{\beta} \\ \frac{\beta\varepsilon}{\delta} & 0 \end{bmatrix}$, whose characteristic polynomial is $\mathcal{P}(\lambda) = \lambda^2 + \varepsilon\gamma$. The eigenvalues for this case are given by $\lambda = \pm\sqrt{\varepsilon\gamma}\, i$ and since we have zero real parts, we have a pure imaginary.

By Theorem 1, system (19) is asymptotically stable if and only if $|\arg[spec(\mathcal{J})]| > \alpha\frac{\pi}{2}$, which is fulfilled if $0 < \alpha < 1$, as shown in Figs. 6–9.

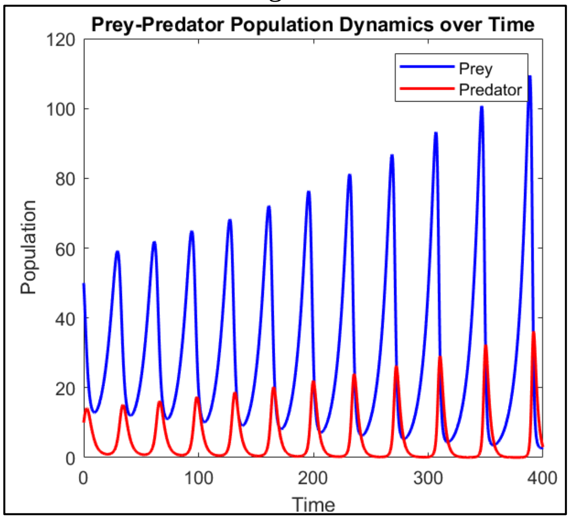

Fig. 6 For $\alpha = 0.9$ (Developed by the authors)

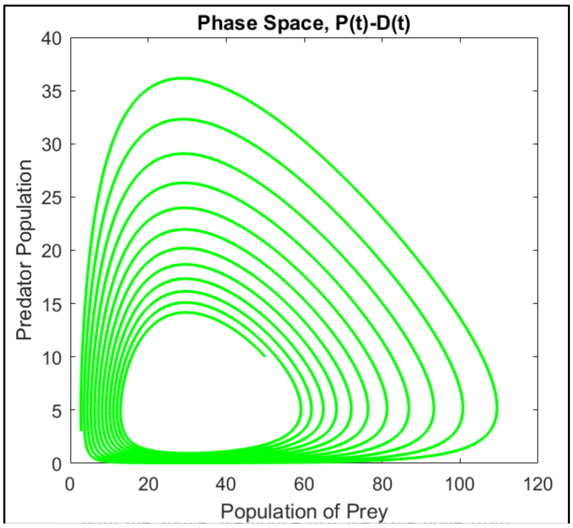

Fig. 7 Trajectories $\alpha = 0.9$ (Developed by the authors)

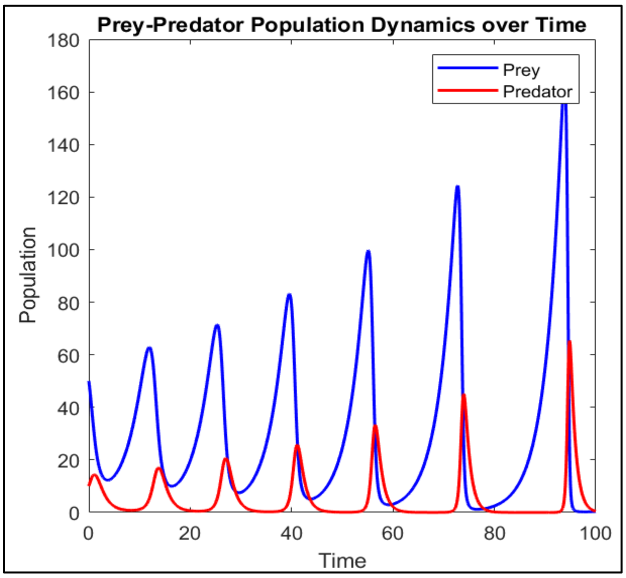

Fig. 8 For $\alpha = 0.5$ (Developed by the authors)

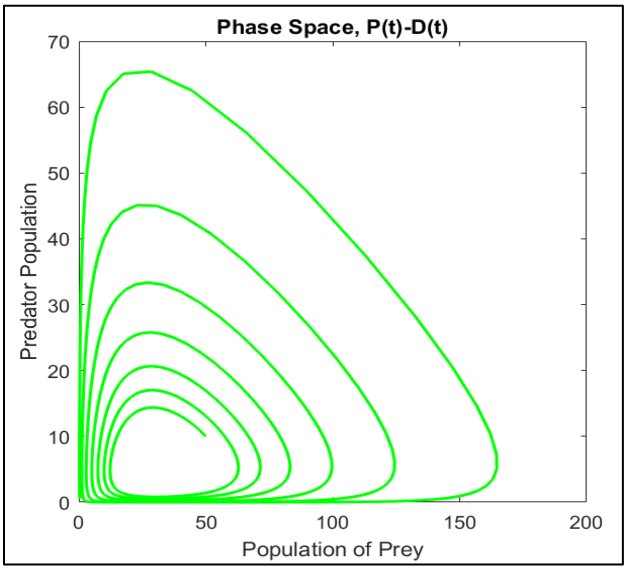

Fig. 9 Trajectories $\alpha = 0.5$ (Developed by the authors)

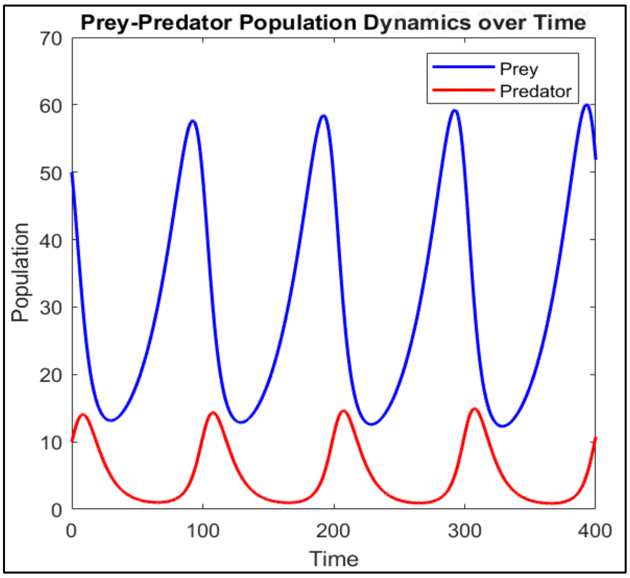

Fig. 10 For $\alpha = 1.3$ (Developed by the authors)

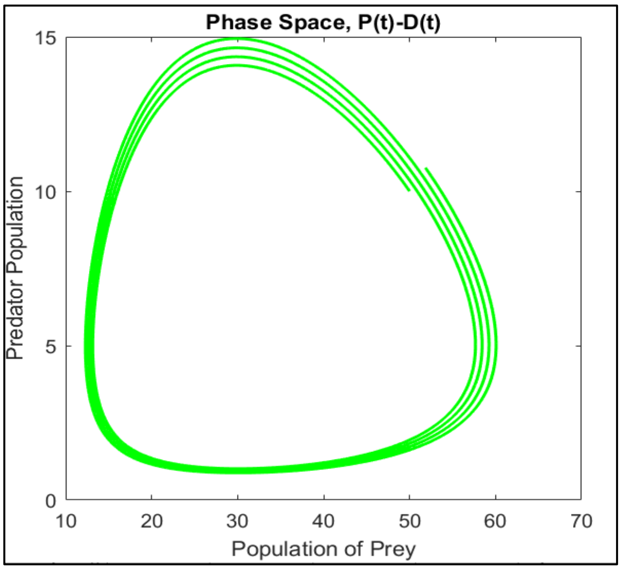

Fig. 11 Trajectories $\alpha = 1.3$ (Developed by the authors)

In Figs. 6 to 11, the initial conditions are set as the values: $P_0 = 50$ as the initial prey population, $D_0 = 10$ as the initial predator population, $\varepsilon = 0.1$ as the intrinsic growth rate of prey, $\delta = 0.02$ as the encounter rate between prey and predators, $\gamma = 0.3$ as the mortality rate of predators in the absence of prey and $\beta = 0.01$ as the reproduction rate of predators per prey consumed.

Figs. 7 and 9 show that the trajectories converge to the equilibrium point, i.e., system (19) is asymptotically stable if $\alpha < 1$. However, if in system (19) $\alpha \geq 1$ is taken, the system has to be stable, but not asymptotically.

We will now use the results of Theorem 11 to analyze the stability of a type of Lotka–Volterra biological model in which three species interact, one acting as prey and the other as predator. Such a system in its fractional version using the derivative operator in the Caputo sense is given by

$$\begin{cases} {}_c\mathfrak{D}^{\alpha}_{0,t}[P(t)] = \eta P - \beta PD \\ {}_c\mathfrak{D}^{\alpha}_{0,t}[D(t)] = -\delta D + \gamma DP - \xi DQ \\ {}_c\mathfrak{D}^{\alpha}_{0,t}[Q(t)] = -\rho Q + \sigma DQ, \end{cases} \qquad (20)$$

for $\eta,\ \beta,\ \delta,\ \gamma,\ \xi,\ \rho,\ \sigma > 0$, where $\eta,\ \beta,\ \delta$ and γ are the parameters of system (18) and ξ represents the effect of predation on species $D(t)$ by species $Q(t)$, ρ represents the natural mortality rate of predator $Q(t)$ in the absence of prey, and σ represents the efficiency and rate of spread of predator $Q(t)$ in the presence of prey.

To analyze the stability of system (20), we first write it in the form $\mathfrak{D}^{\alpha}_{0,t}[x(t)] = Ax(t) + f(x(t))$, and then apply the conditions of Theorem 11. Here, $A = \begin{pmatrix} \eta & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & -\rho \end{pmatrix}$, $f(x(t)) = \begin{pmatrix} -\beta PD \\ \gamma DP - \xi DQ \\ -\rho Q + \sigma DQ \end{pmatrix}$ and $x(t) = \begin{pmatrix} P \\ D \\ Q \end{pmatrix}$.

System (20) is asymptotically stable if it satisfies the following:

1. $|\arg(spec(A))| > \alpha \frac{\pi}{2}$,
2. $\frac{\|f(x(t))\|}{\|x(t)\|} \to 0$ when $\| x(t) \| \to 0$

In fact, the first condition is satisfied if $0 < \alpha < 1$, since the eigenvalues are given by $\lambda_1 = \eta, \lambda_2 = -\delta$ and $\lambda_3 = -\rho$.

We will verify the second condition,

$$\begin{aligned} \frac{\| f(x(t)) \|}{\| x(t) \|} &= \frac{\sqrt{(-\beta PD)^2 + (\gamma DP - \xi DQ)^2 + (\sigma DQ)^2}}{\sqrt{P^2 + D^2 + Q^2}} \\ &\leq \frac{\sqrt{(-\beta PD)^2 + (\gamma DP - \xi DQ)^2 + (\sigma DQ)^2}}{\sqrt{P^2}} \\ &\leq \sqrt{P^2 + D^2 + Q^2} \longrightarrow 0 \; with \; \| x(t) \| \longrightarrow 0 \end{aligned}$$

Therefore, according to Theorem 11, system (20) is asymptotically stable if $\alpha < 1$, because it satisfies the two conditions mentioned above.

In the Figs. 12 to 21, the initial conditions are set as the values $P_0 = 0.5$ as initial prey population, $D_0 = 0.5$ as initial predator/prey population, $Q_0 = 2$ as initial predator population, $\eta = \beta = \delta = \gamma = \xi = \rho = 1$ and we will vary the parameters of σ.

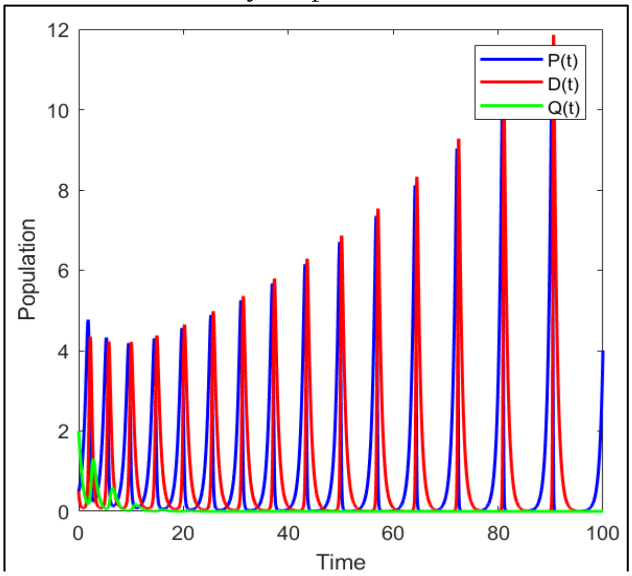

Fig. 12 For $\alpha = 0.9$ and $\sigma = 0.8$ (Developed by the authors)

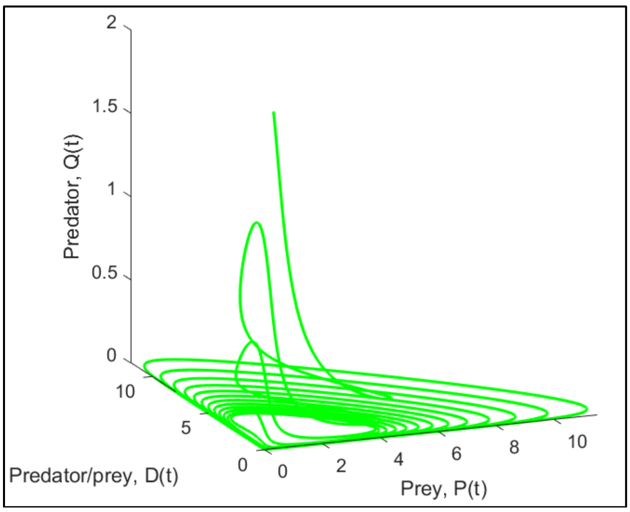

Fig. 13 Trajectories $\alpha = 0.9$ and $\sigma = 0.8$ (Developed by the authors)

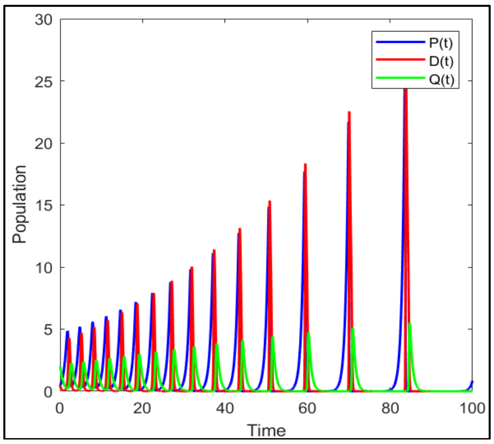

Fig. 14 For $\alpha = 0.9$ and $\sigma = 1.0$ (Developed by the authors)

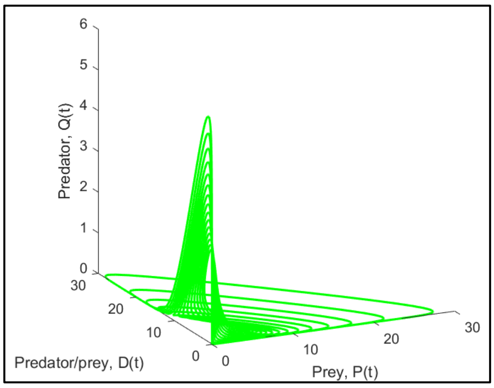

Fig. 15 Trajectories $\alpha = 0.9$ and $\sigma = 1.0$ (Developed by the authors)

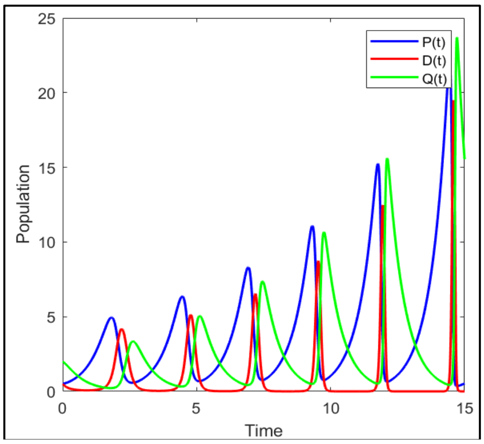

Fig. 16 For $\alpha = 0.9$ and $\sigma = 1.2$ (Developed by the authors)

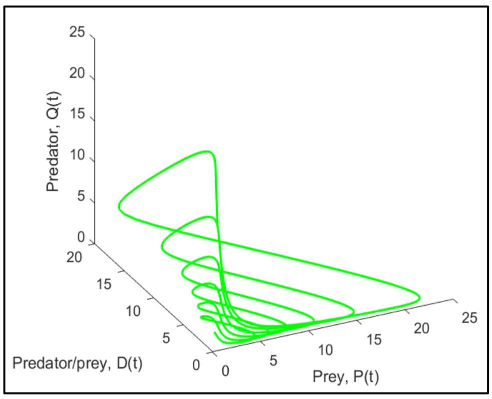

Fig. 17 Trajectories $\alpha = 0.9$ and $\sigma = 1.2$ (Developed by the authors)

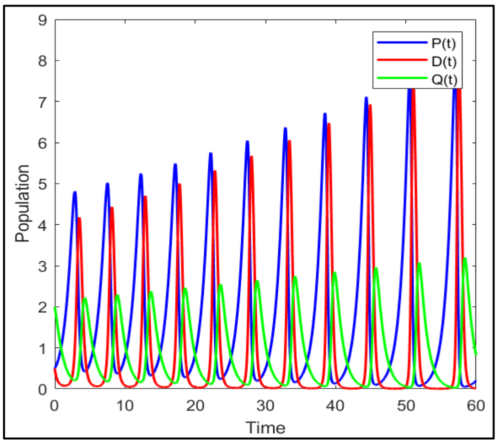

Fig. 18 For $\alpha = 1.0$ and $\sigma = 1.0$ (Developed by the authors)

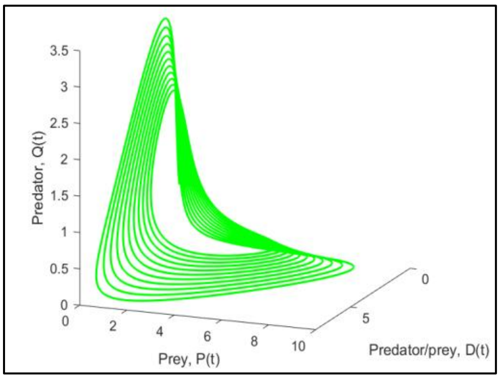

Fig. 19 Trajectories $\alpha = 1.0$ and $\sigma = 1.0$ (Developed by the authors)

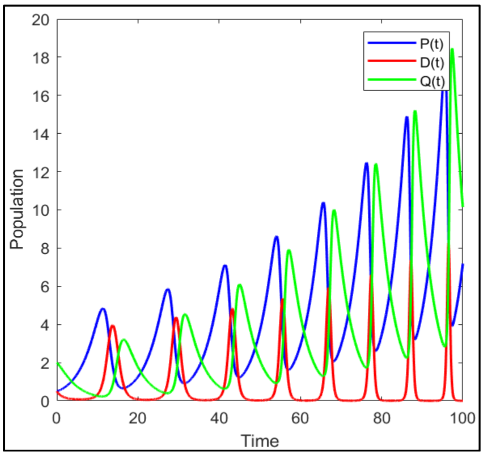

Fig. 20 For $\alpha = 1.3$ and $\sigma = 1.2$ (Developed by the authors)

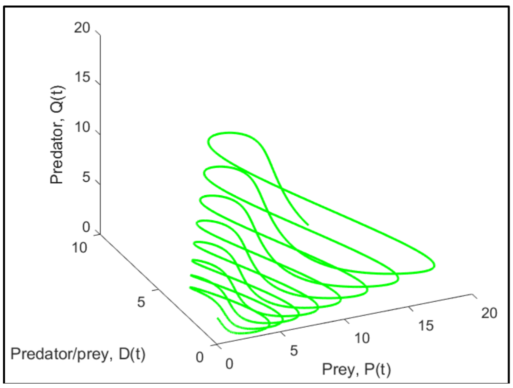

Fig. 21 Trajectories $\alpha = 1.3$ and $\sigma = 1.2$ (Developed by the authors)

In the previous figures, the stability of the system (20) depends both on the order α of the fractional derivative and on the value of the parameters, particularly on the values of σ. The system (20) is asymptotically stable if $\alpha < 1$ and $\sigma < 1$, as shown in Figs. 12 and 13, it is stable if $\alpha = 1$ and $\sigma = 1$, as shown in Figs. 14 to 19, and it is not stable if $\alpha > 1$ and $\sigma > 1$, as shown in Figs. 20 and 21.

5. Conclusion

The study of the stability of systems of ordinary differential equations of fractional order is a fundamental topic of various models of applied sciences and engineering. This paper hopes to collect a large part of the base contributions presented on the stability study from the first results presented by [15] till now. As mentioned before, we apologize if some fundamental references are absent in this research. We demonstrate the primary results within the study of the stability of linear or nonlinear FDEs.

In this study, we observe the significant progress in the construction of a grounded stability theory for FDEs. The stability of linear FDEs is well investigated, and for the stability of nonlinear FDEs, a fundamental theorem is presented and proved. At present, generalizations on the stability concepts for FDE systems are still being presented, supported by the results presented in this paper.

Fundamental results were presented for the stability analysis of the Lotka–Volterra or prey–predator models in their fractional versions. For the case where two species interact, the fractional operators are excellent controllers of the speed with which the trajectories converge toward the respective equilibrium point, thus presenting asymptotic stability, which was not present in the classical model.

For the case where three species interact, fundamental results are presented in the stability study because conditions for which the model is asymptotically stable, stable, and unstable present not only dependence on the order of the fractional derivative operator but also on its involved parameters, which describe the rate of interaction between species.

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