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Qualitative Analysis of Systems of Linear Ordinary Differential Equations of Fractional Order

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Abstract: Nowadays, fractional calculus has been a topic of interest in several areas of applied science and engineering because of the nonlocal character of its fractional operators, which provides additional information that allows a deep analysis of the mathematical models involved. Within fractional calculus, the qualitative approach to generalized ordinary differential equations or fractional order is an open subject of study, in which stability analysis plays a fundamental role in various applied models. This paper aims to present the foundational and recent results of the qualitative study of fractional order linear ordinary differential equations focused on the stability and some analytical methods used. This article presents and describes the fundamental results of the stability of systems of linear fractional order ordinary primary tools that can apply to various models of applied sciences and engineering.

Keywords: fractional analysis, linear fractional differential equations, qualitative analysis, stability of fractional differential equations, linearization method.

分數階線性常微分方程組的定性分析

摘要：如今，分數階微積分已成為應用科學和工程多個領域的熱門話題，因為它的分數運算子具有非局部特徵，它提供了額外的信息，可以對所涉及的數學模型進行深入分析。在分數階微積分中，廣義常微分方程或分數階的定性方法是一個開放的研究主題，其中穩定性分析在各種應用模型中發揮基礎作用。本文旨在介紹分數階線性常微分方程質性研究的基礎和最新結果，重點在於穩定性和所使用的一些分析方法。本文介紹並描述了線性分數階普通主要工具系統穩定性的基本結果，可應用於應用科學和工程的各種模型。

关键词：分數階分析、線性分數階微分方程、質性分析、分數階微分方程的穩定性、線性化方法。

1. Introduction

The invention of fractional calculus, according to some authors, occurred on September 30, 1695, in a letter addressed by L'Hopital to Leibniz arguing for a notation for the derivative of a function in the case where the order was not integer. Despite the antiquity of fractional calculus, only in the last decades have significant developments been presented due to its applicability, in which diverse applied problems obtain better answers in their modeling in its fractional version [1-6].

Currently, most problems in applied sciences and engineering consider fundamental elements of fractional analysis related to the modeling through fractional differential equations. The nonlocal and anomalous nature of problems from these areas of knowledge allows fractional differential equations to model their dynamics.

In addition, fractional analysis has developed mainly due to its applicability in several disciplines of mathematics, including stochastic processes, integral–differential equations, transform theory, numerical analysis, viscoelasticity theory, the study of anomalous diffusion, electromagnetic theory, circuit theory, biology, and the physics of the atmosphere [7-12].

The modeling and analysis of problems applied to the above disciplines can be described using fractional differential equations because of the nonlocal character of their derivative operator, which allows many of the physical phenomena with special memory and genetic characteristics, modeled by these equations [13-24].

Although fractional order differential equations have attracted the attention of researchers, only a few have made significant contributions to constructing a clear and coherent theory of these equations, analogous to the classical case of ordinary differential equations, to support the use of this tool by the applied sciences. Within this study of ordinary differential equations of fractional order, the qualitative approach is an open research topic; it has not had a significant advance that gives rise to the analysis of stability concepts on the dynamics of the solutions of such systems. There is a gap in structured stability theory for this type of differential equation; however, some researchers have presented results from different contexts and focused on stability for fractional ordinary differential equations of linear and nonlinear type.

As mentioned above, there is still no established stability theory for fractional order differential equations; however, in the last two decades, valid contributions have been made to this theory. In 1996, [25], for the first time, analyzed the stability of systems of linear differential equations of fractional type of finite dimension applied to control theory, for these systems presented in the form of state space, demonstrated the so-called internal and external stabilities. He mentioned that the stabilities are guaranteed if the roots of some polynomial lie outside $|\arg(\sigma)| \leq \alpha\pi/2$, thus generalizing the results known for the ordinary case $\alpha=1$ [25]. From that moment on, several researchers became interested in contributing to the stability of systems of fractional differential equations of linear type.

The initial results on the stability of systems of differential equations of the fractional order of linear type, starting from the results proposed by [25], have been generalized for order $0<\alpha< 2$ by some researchers but from different approaches. In [26-28] they study the stability case for the linear system for order $1<\alpha<2$ using tools from control theory, such as linear matrix inequality (LMI) methods, and establish the stability region as a generalization of the results proposed by [25].

This article presents three sections structured as follows: the first section presents the fundamental preliminaries of fractional operators, their properties, and the base concepts of fractional order differential equations; the second section presents the fundamental concepts of stability of systems of linear differential equations. In addition, the linearization method for systems of nonlinear differential equations is presented, and the last section presents a discussion of the results obtained from the research.

2. Preliminaries

In this section, we will initially present the fundamental preliminaries on fractional analysis and then present the relevant concepts for the study of ordinary differential equations of fractional order using derivative operators in the sense of Riemann–Liouville, Caputo, and Hadamard. The core references in this section are taken from $[24, 29-35]$.

In this article, $\mathbb{Z}_+$ denotes the set of positive integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ represents the set of complex numbers, $\mathbb{R}e(\alpha)$ represents the real part of the complex number α and $\Gamma(\cdot)$ represents the Gamma function.

Definition 1: The fractional integral in the *Riemann–Liouville* sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) > 0$) of the function $x(t)$ denoted by $\mathcal{D}_{t_0,t}^{-\alpha}[x(t)]$ or $\mathcal{J}_{t_0,t}^{\alpha}[x(t)]$, is given by:

$$\mathcal{J}_{t_0,t}^{\alpha}[x(t)] = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{x(s)}{(t-s)^{1-\alpha}}\, ds, \qquad t > t_0 \quad (1)$$

Definition 2: The fractional derivative in the *Riemann–Liouville* sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) \geq 0$) of the function $x(t)$ denoted by $\mathcal{D}_{t_0,t}^{\alpha}[x(t)]$, is given by

$$\begin{aligned}\mathcal{D}_{t_0,t}^{\alpha}[x(t)] &= \frac{d^n}{dt^n}\left(\mathcal{J}_{t_0,t}^{n-\alpha}[x(t)]\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{t_0}^{t} \frac{x(s)}{(t-s)^{\alpha+1-n}}\, ds, \quad t > t_0,\end{aligned} \quad (2)$$

where $n-1 < \alpha < n$, with $n \in \mathbb{Z}_+$.

Caputo's fractional derivative allows the consideration of initial conditions of natural order to mathematical models represented by fractional differential equations. These fractional derivative operators are defined as follows:

Definition 3: The fractional derivative in the Caputo sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) \geq 0$) of the function $x(t)$ denoted by ${}_c\mathfrak{D}^{\alpha}_{t_0,t}[x(t)]$, is given by

$$\begin{aligned} {}_c\mathfrak{D}^{\alpha}_{t_0,t}[x(t)] &= \mathcal{J}^{n-\alpha}_{t_0,t}\left[x^{(n)}(t)\right] \\ &= \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{x^{(n)}(s)}{(x-s)^{\alpha+1-n}}\,ds, t > t_0, \end{aligned} \quad (3)$$

where $x^{(n)}(s)$ represents the nth derivative of the function $x(s)$ and $n-1<\alpha<n$ with $n \in \mathbb{Z}_+$.

Another fundamental definition of fractional operators is the fractional derivative and integral in Hadamard's sense.

Definition 4: The fractional integral in the Hadamard sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha)>0$) of function $x(t)$ denoted by ${}_{\mathcal{H}}\mathfrak{D}^{-\alpha}_{t_0,t}[x(t)]$ or ${}_{\mathcal{H}}\mathcal{J}^{\alpha}_{t_0,t}[x(t)]$, is given by

$${}_{\mathcal{H}}\mathcal{J}^{\alpha}_{t_0,t}[x(t)] = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{x(s)}{s}\,ds, t > t_0 \quad (4)$$

Definition 5: The fractional derivative in the *Hadamard* sense of order α (with $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha)\geq 0$) of the function $x(t)$ denoted by ${}_{\mathcal{H}}\mathcal{D}^{\alpha}_{t_0,t}[x(t)]$, is given by

$${}_{\mathcal{H}}\mathfrak{D}^{\alpha}_{t_0,t}[x(t)] = \left(t\frac{d}{dt}\right)^{n} {}_{\mathcal{H}}\mathcal{J}^{n-\alpha}_{t_0,t}\left[x^{(n)}(t)\right]$$

$$= \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^{n}\int_{t_0}^{t}\left(\ln\frac{t}{s}\right)^{n-\alpha+1}\frac{x(s)}{s}\,ds, t > t_0 \quad (5)$$

where $n-1<\alpha<n$, with $n \in \mathbb{Z}_+$.

Just as in the solutions of ordinary differential equations of integer order, the exponential function plays a fundamental role. In addition, in the solutions of differential equations of fractional order, an analogous function called the Mittag–Leffler function is used.

Definition 6: The Mittag–Leffler function of a parameter α is given by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},$$

where $z \in \mathbb{C}$, $\mathbb{R}e(\alpha)>0$. The Mittag–Leffler function of two parameters α and β is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

where $z,\beta \in \mathbb{C}$, $\mathbb{R}e(\alpha)>0$.

When $\beta=1$, $E_{\alpha,\beta}(z)$ coincides with the one-parameter Mittag–Leffler function, that is, $E_{\alpha,1}(z)=E_{\alpha}(z)$. Moreover, when $\alpha=\beta=1$, the function $E_{1,1}(z)$ is equivalent to the function e^z.

Proposition 1: If $0<\alpha<2$ and $\beta \in \mathbb{C}$ arbitrary, then for a $n \in \mathbb{Z}$ with $n \geq 1$ we have the following expansions:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}z^{\frac{1-\beta}{\alpha}}e^{z^{\frac{1}{\alpha}}} - \sum_{k=1}^{n}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k} + O\left(\frac{1}{|z|^{n+1}}\right),$$

with $|z|\to\infty$, $|\arg(z)|\leq \alpha\frac{\pi}{2}$, and

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{n}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k} + O\left(\frac{1}{|z|^{n+1}}\right),$$

with $|z|\to\infty$, $|\arg(z)|>\alpha\frac{\pi}{2}$.

The following results can be considered generalizations of the exponential function.

Definition 7: The function $e^{\lambda z}_{\alpha} = z^{\alpha-1}E_{\alpha,\alpha}(\lambda z^{\alpha})$ is called α −exponential function, where$z \in \mathbb{C}\backslash\{0\}, \lambda \in \mathbb{C}$ and $\mathbb{R}e(\alpha)>0$.

According to Proposition 1 and Definition 6, it follows that $E_{1,1}(z)=E_1(z)$ and indeed $e^{\lambda z}_1 = e^{\lambda z}$.

From definition 7 of $\alpha-$ exponential function and proposition 1, we have the following result:

Proposition 2: If $0<\alpha<2$ and $z \in \mathbb{C}$ then one has the following asymptotic equivalences for the α −exponential function:

For $|\arg(z)|\leq\alpha\frac{\pi}{2}$, you must $e^{\lambda z}_{\alpha} \sim \frac{\lambda^{\frac{1-\alpha}{\alpha}}}{\alpha}e^{\lambda^{\frac{1}{\alpha}}}z$ when $|z|\to\infty$, for $|\arg(z)|>\alpha\frac{\pi}{2}$, you must $e^{\lambda z}_{\alpha} \sim -\frac{1}{\lambda^2\Gamma(-\alpha)}\frac{1}{z^{\alpha+1}}$ when $|z|\to\infty$.

The following are the fundamental definitions of ordinary differential equations of fractional order.

Definition 8: An ordinary differential equation of fractional order α is defined as follows:

$$\mathbb{D}^{\alpha}_{t_0,t}[x] = f\left(t,x,\mathbb{D}^{\alpha_1}_{t_0,t}[x],\mathbb{D}^{\alpha_2}_{t_0,t}[x],\ldots,\mathbb{D}^{\alpha_{m-1}}_{t_0,t}[x]\right) \quad (6)$$

where $x(t)$ is a real domain complex unknown function, $f(t,x,x_1,x_2,x_3,\ldots,x_{n-1})$ is a known function and $\mathbb{D}^{\alpha_k}_{t_0,t}$ for $k=1,2,3,\ldots,m-1$ are fractional differential operators such that $0<\mathbb{R}e(\alpha_1)<Re(\alpha_2)<Re(\alpha_3)<\cdots<Re(\alpha_{m-1})<Re(\alpha)$ for $m \in \mathbb{Z}_+, m\geq 2$.

Definition 9: An ordinary differential equation of fractional order α of Linear type is defined as follows:

$$\mathbb{D}^{\alpha}_{t_0,t}[x] + a_0(t)x + \sum_{k=1}^{m-1}a_k(t)\mathbb{D}^{\alpha_k}_{t_0,t}[x] = f(t) \quad (7)$$

where $x(t)$ is a real domain complex unknown function, $a_k(t)$ for $k=0,1,2,3,\ldots,m-1$ and $f(t)$ are known functions, $\mathbb{D}^{\alpha}_{t_0,t}$ for $k=1,2,3,\ldots,m-1$ are fractional differential operators such that $0<\mathbb{R}e(\alpha_1)<Re(\alpha_2)<Re(\alpha_3)<\cdots<Re(\alpha_{m-1})<Re(\alpha)$ for $m \in \mathbb{Z}_+, m\geq 2$.

If the functions $a_k(t)$ are constant for all $k=0,1,2,3,\ldots,m-1$, equation (7) is said to have constant coefficients. On the contrary, if at least one of them is variable, the equation is said to have variable coefficients. Now, if $f(t)=0$, the linear fractional differential equation of the ordinary type is said to be homogeneous.

The Cauchy-type problem for the Riemann–Liouville fractional derivative presented in equation (1) is given by

$$\begin{cases} \mathcal{D}^{\alpha}_{t_0,t}[x(t)] &= f\left(t,x,\mathcal{D}^{\alpha_1}_{t_0,t}[x],\mathcal{D}^{\alpha_2}_{t_0,t}[x],\ldots,\mathcal{D}^{\alpha_{m-1}}_{t_0,t}[x]\right) \\ \mathcal{D}^{\alpha-k}_{t_0,t}[x(t)]|_{t=t_0} &= b_k, \quad b_k \in \mathbb{C}, \quad k=1,2,3,\ldots,m. \end{cases} \quad (8)$$

where $m = \begin{cases} \lfloor \mathbb{R}e(\alpha) \rfloor + 1 & if \quad \alpha \notin \mathbb{Z}_+ \\ \alpha & if \quad \alpha \in \mathbb{Z}_+ \end{cases}$.

The Cauchy problem for the Caputo fractional derivative presented in equation (2) has the following structure:

$$\begin{cases} {}_{c}\mathfrak{D}^{\alpha}_{a^+,t}[x] & = f\left(t, x, {}_{c}\mathfrak{D}^{\alpha_1}_{a^+,t}[x], {}_{c}\mathfrak{D}^{\alpha_2}_{a^+,t}[x], \dots, {}_{c}\mathfrak{D}^{\alpha_{m-1}}_{a^+,t}[x]\right) \\ x^{(k)}(a) & = b_k, \quad b_k \in \mathbb{C}, \quad k = 1,2,3,\dots,m. \end{cases} \quad (9)$$

where $0 < \mathbb{R}e(\alpha_1) < Re(\alpha_2) < Re(\alpha_3) < \cdots < Re(\alpha_{m-1}) < Re(\alpha)$ for $m \in \mathbb{Z}_+, m \geq 2$.

In this case, the initial conditions are given in terms of ordinary derivatives, which facilitate the physical interpretation.

The ordinary differential equation of fractional order presented in Equation (6) can be expressed as follows:

$$\mathbb{D}^{\tilde{\alpha}}_{t_0,t}[x(t)] = f\big(t, x(t)\big), \quad (10)$$

where $f: [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\tilde{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_n]^T$ with $m - 1 < \alpha_i < m$ for $m \in \mathbb{Z}_+$ and $i = 1,2,\dots,n$, $x_k = [x_{k_1}, x_{k_2}, \dots, x_{k_n}]^T \in \mathbb{R}^n$ for $k = 0,1,2,\dots,m-1$ represent suitable initial conditions where $x(t) = [x_1(t), x_2(t) \dots, x_n(t)]^T \in \mathbb{R}^n$. The derivative operator denoted by $\mathbb{D}^{\tilde{\alpha}}_{t_0,t}$ represents the derivative in the Caputo or Riemann–Liouville sense.

If $\alpha_1, \alpha_2, \dots, \alpha_n = \alpha$ then Equation (10) can be written as

$$\mathbb{D}^{\alpha}_{t_0,t}[x(t)] = f\big(t, x(t)\big), \quad (11)$$

and is called a system of fractional differential equations of the same order.

Definition 10: A constant vector x_* is said to be the equilibrium point of the system of fractional differential equations (10), if and only if $f(t, x_*) = \mathbb{D}^{\tilde{\alpha}}_{t_0,t}[x(t)]|_{x(t)=x_*}$ for all $t > t_0$.

Without loss of generality, we can consider the equilibrium point at the origin, that is, $x_* = 0$ and establish the following definition.

Definition 11: The solution $x_* = 0$ of the system of fractional order differential equations (11), is given by

1. Stable, if for any initial condition $x_k = [x_{k_1}, x_{k_2}, \dots, x_{k_n}]^T \in \mathbb{R}^n$, with $= 0,1,2,\dots,m-1$, there exists $\epsilon > 0$ such that any solution $x(t)$ of equation (11) satisfies that$||x(t)|| < \epsilon$ for all $t > t_0$.

2. Asymptotically stable if it is stable and is satisfied that $||x(t)|| \to 0$ when $t \to +\infty$.

3. Stability of Linear Fractional Ordinary Differential Equations (LFDEs)

In this section, we consider the main results that have been presented on the stability of systems of ordinary differential equations of fractional order of linear type. For this, we consider the linear system of FDEs

$$\mathbb{D}^{\tilde{\alpha}}_{t_0,t}[x(t)] = Ax(t), \quad (12)$$

where $x(t) = [x_1(t), x_2(t) \dots, x_n(t)]^T \in \mathbb{R}^n$, A is a matrix such that $A \in \mathbb{R}^{n \times n}$, $\tilde{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_n]^T$, $\mathbb{D}^{\tilde{\alpha}}_{t_0,t}[x(t)] = \left[\mathbb{D}^{\alpha_1}_{t_0,t}[x_1(t)], \mathbb{D}^{\alpha_2}_{t_0,t}[x_2(t)], \dots, \mathbb{D}^{\alpha_n}_{t_0,t}[x_n(t)]\right]^T$.

The derivative operator by notation $\mathbb{D}^{\alpha_i}_{t_0,t}$ represents the derivative in the sense of Caputo or Riemman–Liouville of order α_i, where $0 < i \leq 2$ for $i = 1,2,\dots,n$. If $\alpha_1, \alpha_2, \dots, \alpha_n = \alpha$ then the Equation (12) can be written as

$$\mathbb{D}^{\alpha}_{t_0,t}[x(t)]Ax(t). \quad (13)$$

The first stability results for linear FDE systems were presented using tools from an algebraic approach and using asymptotic results applied to control theory [25]. Said result on stability was established for the system (13) for the order $0 < \alpha \leq 1$, which is presented below.

Theorem 1: The autonomous system (13) with the Caputo derivative and initial value $x_0 = x(0)$, where $0 < \alpha \leq 1$, is:

1. Asymptotically stable if and only if $|\arg[spec(A)]| > \alpha\frac{\pi}{2}$. In this case, the state components decay toward 0 as $\frac{1}{t^{\alpha}}$.

2. Stable if and only if either it is asymptotically stable or those critical eigenvalues which satisfy $|\arg[spec(A)]| = \alpha\frac{\pi}{2}$ have geometric multiplicity one, where $spec(A)$ denotes the eigenvalues of the matrix A corresponding to system (13).

From the classical theory of stability for systems of ordinary differential equations of the linear type of integer order, we know that we can establish the stability of a linear system at its equilibrium point by studying the eigenvalues of the matrix associated A to the system. This algebraic study establishes that if the roots of the characteristic polynomial associated with the matrix A have a negative real part, then the linear system is asymptotically stable.

The result presented by [25] in Theorem 1 for the case where $0 < \alpha < 1$, shows that the roots of the characteristic polynomial associated with the matrix A of the system (13) lie outside the closed angular sector $|\arg[spec(A)]| \leq \alpha\frac{\pi}{2}$, as shown in Fig. 1.

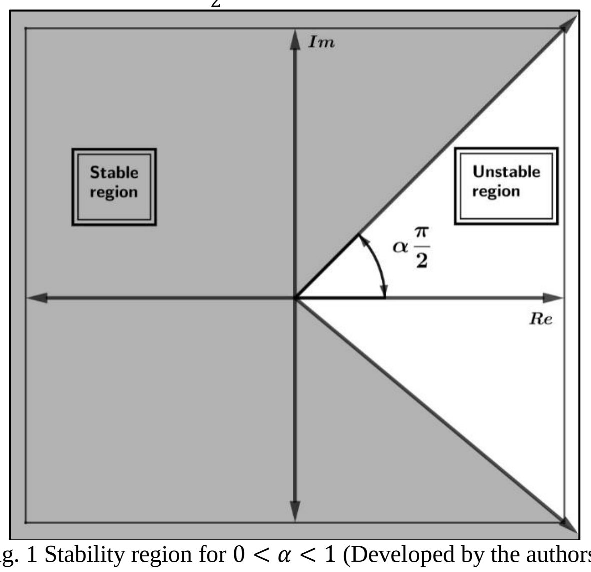

Fig. 1 Stability region for $0 < \alpha < 1$ (Developed by the authors)

For the case where $\alpha = 1$ we have the stability region presented in the classical sense of linear ordinary differential equations of natural order, as shown in Fig. 2.

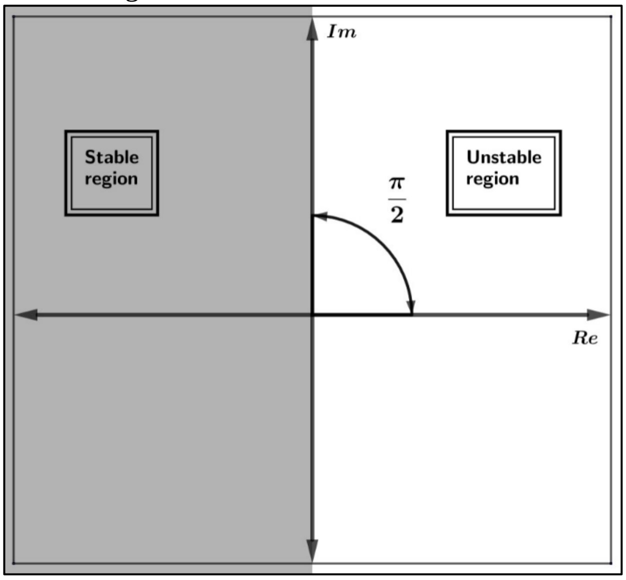

Fig. 2 Stability region for $\alpha = 1$ (Developed by the authors)

Note that the asymptotic stability of the system (13) presented in Theorem 1 is also called $t^{-\alpha} -$ stability because the state components exhibit anomalous decay.

On the other hand, [36] proved Theorem 1 without using control theory entities and for the non-asymptotic stability of the system (13) and presented the following theorem.

Theorem 2: If all the eigenvalues of the matrix A satisfy $|\arg[spec(A)]| \geq \alpha\frac{\pi}{2}$ and the critical eigenvalues satisfying $|\arg[spec(A)]| = \alpha\frac{\pi}{2}$ have the same algebraic and geometric multiplicities, then the zero solution of system (13) is stable but not asymptotically stable.

Furthermore, for system (13), [43] studied the case when there are null eigenvalues in the matrix A of the linear system using the Riemann–Liouville derivative operator with asymptotic expansions of the Mittag–Leffler function for the order $0 < \alpha < 1$, of the following way [36].

Theorem 3: The system (13) with the Riemann–Liouville derivative and initial value $x_0 = \mathcal{D}_{t_0,t}^{\alpha-1}[x(t)]|_{t=t_0}$, where $0 < \alpha < 1$ and $t_0 = 0$, is

1. Asymptotically stable if and only if all non- zero eigenvalues of the matrix A satisfy $|\arg[spec(A)]| > \alpha\frac{\pi}{2}$, or the A has k multiple zero eigenvalues corresponding to Jordan block $diag(J_1, J_2, \dots, J_i)$, where J_l is a Jordan canonical form with order n_l, for $\sum_{l=1}^{i} n_l = k$, and $n_l\alpha < 1$, for $1 \leq l \leq i$.
2. Stable if and only if either it is asymptotically stable, those critical eigenvalues which satisfy $|\arg[spec(A)]| = \alpha\frac{\pi}{2}$ have the same algebraic and geometric multiplicities, or the A has k multiple zero eigenvalues corresponding to a Jordan block $diag(J_1, J_2, \dots, J_i)$, where J_l is a Jordan canonical form with order n_l, for $\sum_{l=1}^{i} n_l = k$, and $n_l\alpha < 1$, for $1 \leq l \leq i$.

In the above theorem, the state components decay toward 0 as $t^{-\alpha-1}$ if all eigenvalues of the system matrix A satisfy $|\arg[spec(A)]| > \alpha\frac{\pi}{2}$. If all the nonzero eigenvalues of the matrix A satisfy $|\arg[spec(A)]| \geq \alpha\frac{\pi}{2}$ and the critical eigenvalues that satisfy $|\arg[spec(A)]| = \alpha\frac{\pi}{2}$ have the same algebraic and geometric multiplicities, and the zero eigenvalue of matrix A has the same algebraic and geometric multiplicities, then the null solution of system (13) is stable from the representation of the solution.

For the multiple-order linear fractional system of differential equations, as expressed in Equation (13), presented the following output [37].

Theorem 4: Suppose that α_i's are rational numbers between 0 and 1, for $i = 1,2, \dots, n$. Let M be the lowest common multiple (LCM) of the denominators u_i of α_i's, where $\alpha_i = \frac{v_i}{u_i}$,$(u_i, v_i) = 1$, u_i, $v_i \in \mathbb{Z}_+, i = 1,2, \dots, n$, and set $\gamma = \frac{1}{M}$. Then, the zero solution of system (1) with the Caputo derivative and initial value $x_0 = x(0)$ is:

1. Asymptotically stable if and only if any zero solution of the polynomial $\det[diag(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \dots, \lambda^{M\alpha_n}) - A]$satisfies $|\arg(\lambda)| > \gamma\frac{\pi}{2}$, the components of the state variable $[x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n$ decay toward 0 like $t^{-\alpha_1}, t^{-\alpha_2}, \dots, t^{-\alpha_n}$, respectively.
2. Stable if and only if either it is asymptotically stable or the critical zero solutions λ of the above polynomial satisfy $|\arg(\lambda)| > \gamma\frac{\pi}{2}$ have geometric multiplicity one.

Note that if $\alpha_1, \alpha_2, \dots, \alpha_n = \alpha$ are rational numbers for $0 < \alpha_i < 1$ with $i = 1,2, \dots, n$, then Theorem 4. coincides with Theorem 1 presented by [25]; that is, the previous theorem is an extension of Theorem 1. with respect to rational orders.

[38], like [39], also analyzed the stability of the system [13] for the case $0 < \alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_n \leq 1$ using Mittag–Leffler functions and their integer order derivatives to obtain analytic solutions of the initial value problem (13) and then establish the condition of sufficient stability using the final value theorem [38].

All the above conclusions refer to the case of commensurable fractional order. Moreover, in [39] they also study the case of incommensurable fractional order. If $\alpha_1, \alpha_2, \dots, \alpha_n$ are irrational numbers between 0 and 1 in the system of differential equations (13), we obtain the following result.

Theorem 5: If all the roots of the characteristic equation $\det[diag(s^{\alpha_1}, s^{\alpha_2}, \dots, s^{\alpha_n}) - A] = 0$ have negative real parts, then the zero solution of system

(13) is asymptotically stable, where α_i is real and lies in (0,1).

From the previous theorems and Proposition 1, we obtain the following result.

Theorem 6: The autonomous same order system (13) with initial value $x_0 = x(0)$ and Riemann–Liouville derivative is asymptotically stable if and only if $|\arg[spec(A)]| > \alpha\frac{\pi}{2}$, where $n = 2$ and $0 < \alpha \leq 1$.

The proof of the previous theorem can be consulted in [40], which uses from [29] the unique solution of the system (13) expressed as a generalization of the matrix α −exponential.

The initial results on the stability of linear fractional order differential equation systems, based on the results proposed by [25], have been generalized to the order $0 < \alpha < 2$ by some researchers but from different approaches. In $[26 - 28]$, they study the stability case for the linear system (13) for the order $1 < \alpha < 2$ using control theory tools such as linear matrix inequality (LMI) methods and establish the region of stability as a generalization of the results proposed by [25].

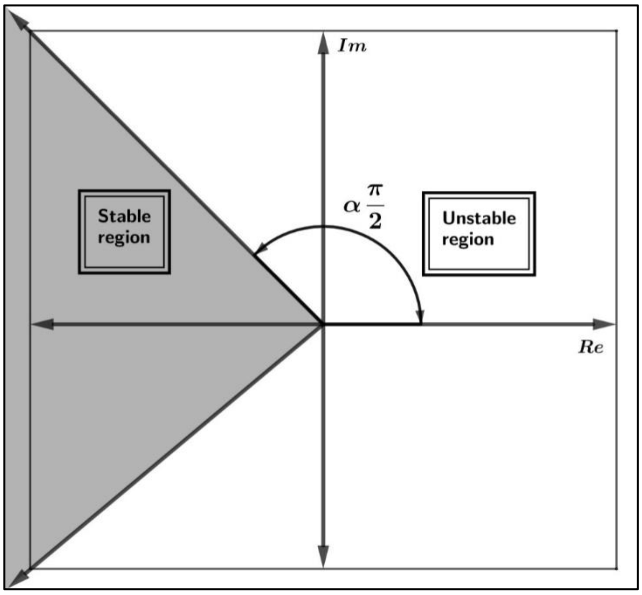

Fig. 3 Stability region for $1 < \alpha < 2$ (Developed by the authors)

Indeed, for the case where $\alpha = 1$, we have the region of stability that we know in the classical sense of the derivative of integer order, as shown in Fig. 2.

In [41], the stability result for the order $1 < \alpha < 2$ is also generalized by performing an extension of Theorem 1., to obtain the following result.

Theorem 7: The autonomous same order system (13) with Riemann–Liouville derivative and initial values $x_k = \mathcal{D}_{0,t}^{\alpha-k-1}[x(t)]|_{t=0}$, for $k = 0,1$, is asymptotically stable if and only if $|\arg[spec(A)]| > \alpha\frac{\pi}{2}$, where $n = 2$ and $1 < \alpha < 2$.

The proof can be found in [29] or [41]. See [42] and [43] where the stability of linear FDE systems is studied using various techniques for stability analysis.

In [44], the authors discuss the stability of the following linear FDE system with the Riemann–Liouville derivative operator is expressed as

$$\begin{cases} \mathcal{D}_{0,t}^{\alpha}[x(t)] & = Ax(t) + B(t)x(t) \\ \mathcal{D}_{0,t}^{\alpha-1}[x(t)]|_{t=0} & = x_0, \end{cases} \quad (14)$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^n \times \mathbb{R}^n$ is a matrix and $B(t): [0,\infty] \to \mathbb{R}^n \times \mathbb{R}^n$ is a continuous matrix. For system (3), sufficient conditions are given to establish stability and asymptotic stability using the Mittag–Leffler function, the generalized Gronwall inequality and the comparison principle for orders $0 < \alpha < 1$ and $1 < \alpha < 2$.

In [45], the authors conduct a detailed analysis of the stability of systems of linear FDEs using the derivative operator in Caputo's sense. To do so, they initially present a new proof of Theorem 1, proposed by [25], to later study the case of systems of linear FDEs of multiple orders.

3.1. On the Linearization Theorem of FDEs

Let us recall that the linearization method is used to study the stability of the local type of an equilibrium point of a given system of nonlinear ordinary differential equations in the case of integer order.

In addition, the linearization method allows the use of tools that are used for the study of systems of linear ordinary differential equations in the analysis of the behavior of systems of nonlinear ordinary differential equations around a given point. It is noteworthy that in [46], the authors presented the linearization theorem of a system of ordinary differential equations of integer order.

The generalization of the linearization theorem for the case of FDEs is much more complex because the fractional differential operators are non-local and have weak singular kernels. However, some researchers initially proposed a method similar to linearization to analyze the stability of the EDF equilibrium points, but without presenting a rigorous proof of said theorem $[47, 48, 49]$. They mentioned the following:

Given $\alpha \in (0,1]$ and the initial value problem

$$ {}_{c}\mathfrak{D}_{t_0,t}^{\alpha}[x(t)] = f\big(t, x(t)\big) \quad (15)$$

$$x(t_0) = x_0$$

Let x_* be an equilibrium point of (15), that is ${}_{c}\mathfrak{D}_{0,t}^{\alpha}[x(t)]|_{x(t)=x_*} = f(x_*) = 0$. To establish asymptotic stability, let $x(t) = x_* + \varepsilon(t)$ and substituting this term in equation (15), we have

$${}_{c}\mathfrak{D}_{0,t}^{\alpha}[\varepsilon(t)] = f\big(x_* + \varepsilon(t)\big),$$

but

$$f\big(x_* + \varepsilon(t)\big) \simeq f(x_*) + f'(x_*)\varepsilon(t) + \cdots$$

$$\Rightarrow f\big(x_* + \varepsilon(t)\big) \simeq f(x_*)\varepsilon(t)$$

and thus we obtain a linear system with its respective initial conditions

$${}_{c}\mathfrak{D}_{t_0,t}^{\alpha}[\varepsilon(t)] = A\,\varepsilon(t) \quad (16)$$

$$\varepsilon(0) = x_0 - x_*,$$

where $A = \frac{\partial f}{\partial x}\Big|_{x_*}$.

With the above, we notice that we have gone from the nonlinear FDE system (15) to the linear FDE system (16). Using the result presented by [25] in Theorem 1, then if $|\arg[spec(A)]| > \alpha\frac{\pi}{2}$, $\varepsilon(t)$ is decreasing and indeed the equilibrium point x_* is locally asymptotically stable. Some researchers linearized a system of non-linear FDEs to establish the stability of certain mathematical models $[50, 51]$, but without specifying the proof of the theorem in question.

Already in $[52]$, the researchers first present some results for the fractional dynamical system defined by the fractional differential equation with Caputo derivative and show the linearization and stability theorems of the system of nonlinear fractional-order differential equations.

For this, the initial value problem of the system of fractional differential equations with the Caputo derivative was initially considered:

$$ {}_{C}\mathfrak{D}_{t_0,t}^{\alpha}[x(t)] = f\big(x(t)\big) \quad (17)$$
$$x(t_0) = x_0,$$

where $0 < \alpha < 1$, $f(x) = [f_1(x), f_2(x), \ldots, f_n(x)]^T$, $x(t) \in \Omega \subset \mathbb{R}^n$, $t \in \mathbb{R}^+$ and the following results are presented for (17).

Lemma 1: If function $f(x)$ is continuous, the initial value problem (17) is equivalent to the following nonlinear Volterra integral equation of the second kind:

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(x(\tau))d\tau \quad (18)$$

Theorem 8: Let $f(x)$ be a continuous function and $x(t)$ be the continuous solution of Eq. (18), then there exists a $\phi(t)$ which satisfies the following properties:

1. $\phi_0 = Id$.
2. $\phi_{t+s} = \phi_t \circ \theta_t \circ \phi_s$, for $s, t \in \mathbb{R}^+$, where θ_t is a linear map satisfying:

$$\theta_t \circ \phi_s(x_0) = x_0 \frac{1}{\Gamma(\alpha)}\int_0^s (t+s - \tau)^{\alpha-1} f(\phi_\tau(x_0))d\tau, \quad t \geq 0,$$

and if $s = 0$, then $\theta_t(x_0) = x_0$.

3. $(t, x_0) \to \phi_t(x_0)$ gives a continuous map from $\mathbb{R}^+ \times \Omega$ onto Ω.

Definition 12: ϕ_t which satisfies $(1) - (3)$ is called a fractional flow in the Caputo sense, and $\{\mathbb{R}^+, \Omega, \phi_t\}$ is a fractional dynamical system in the Caputo sense.

Consider the homogeneous linear systemas follows:

$$\mathbb{D}_{0,t}^{\alpha}[x(t)] = Ax(t). \quad (19)$$

where A is an $n \times n$ constant matrix, $x(0) = x_0$, $0 < \alpha < 1$, and $x(t) \in \mathbb{R}^n$.

Definition 13: If all eigenvalues $\lambda(A)$ of A of the system (19) satisfy: $|\lambda(A)| \neq 0$ and $|\arg(\lambda(A))| \neq \alpha\frac{\pi}{2}$, then the origin 0 of the linear system (19) is called a hyperbolic equilibrium point.

Definition 14: Suppose that x_* is an equilibrium point of system (17) and that all the eigenvalues $\lambda(Df(x_*))$ of the linearized matrix $Df(x_*)$ at the equilibrium point x_* satisfy: $|\lambda(Df(x_*))| \neq 0$ and $|\arg(\lambda(Df(x_*)))| \neq \alpha\frac{\pi}{2}$, then we call x_* a hyperbolic equilibrium point.

Suppose $f(x)$ and $g(y)$ are continuous vector fields defined on $U, V \subseteq \mathbb{R}^n$, and they generate flows $\phi_{t,f}: U \to U$ and $\phi_{t,g}: V \to V$.

Definition 15: If there is a homeomorphism $h: U \to V$, satisfying that $(h \circ \phi_{t,f})(x) = (\phi_{t,g} \circ h)(x)$, for $x \in \delta(x_0, r) \subset U$, $x_0 \in U$, then $f(x)$ and $g(y)$ are locally topologically equivalent. If the above relation holds in the whole space U, they are globally topologically equivalent.

Next, we present the linearization theorem of the fractional differential equation with a derivative in Caputo's sense.

Theorem 9: If the origin 0 is a hyperbolic equilibrium point of (17), then vector field $f(x)$ is topologically equivalent to its linearization vector field $Df(0)x$ in the neighborhood $\delta(0)$ of the origin 0.

To emphasize, the previous theorem can be considered as the fractional version of Hartmans theorem $[46]$. In addition, the condition of hyperbolic equilibrium is necessary. If the origin 0 is not a hyperbolic equilibrium, the conclusion does not hold. Theorem's proof 9. can be consulted in $[52]$. Also in $[53, 54, 55]$ the stability of systems of nonlinear fractional differential equations is studied using the linearization theorem.

In $[56]$, the linearization theorem is generalized to nonlinear fractional systems of differential equations involving the Riemann– Liouville derivative and Hadamard with different types of initial value conditions, noting that these initial value conditions are not equivalent to each other.

Having established the linearization theorem to analyze the stability of nonlinear fractional differential equation systems, techniques have also been generalized to establish the asymptotic stability of both linear and nonlinear FDE systems.

Recall that we can analyze a system of both linear and nonlinear integer order ordinary differential equations using Hurwitz- type polynomials. In this way, the problem of analyzing stability becomes an algebraic type problem because, according to the study of linear differential equations, it is enough to know the roots of the characteristic polynomial associated with the matrix, which correspond to the eigenvalues, and observe if these have a negative real part. If the above occurs, the system is said to be asymptotically stable. If this polynomial has the characteristics mentioned above, it is said to be a Hurwitz polynomial $[57, 58, 59]$.

$[60 - 62]$ studied the Routh–Hurwitz stability criterion to establish Hurwitz-type polynomials and thus analyze the stability of FDE systems for the order of $0 < \alpha < 2$.

$[63 - 65]$ used the Hurwitz criterion to establish the stability of various systems of fractional differential equations from various models applied to various branches of science.

4. Conclusion

The study of the stability of systems of ordinary differential equations of fractional order is a fundamental topic of various models of applied sciences and engineering. This article hopes to collect a large part of the base contributions presented on the stability study from the first results presented by [25] till now. As mentioned before, we apologize if some references are absent in this research. We demonstrate the foundational results within the study of the stability of linear FDEs.

In this study, we observe significant progress in constructing a grounded stability theory for FDEs. The stability of linear FDEs has been well investigated, presenting the linearization method for the stability of nonlinear FDEs. At present, generalizations about the concepts of stability for FDE systems continue to be presented. We hope his research will be helpful for studies of the stability of FDEs.

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