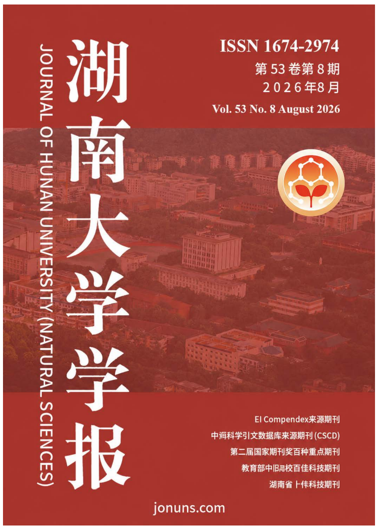

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Graceful Labeling of the Upstairs Graph

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Abstract: Determining whether an arbitrary graph admits a graceful labeling remains an open problem in graph theory, as no general characterization of graceful graphs has been established. This study introduces the Upstairs graph, denoted by (Us_n), as a new graph structure based on a layered construction of grid graphs. Its graceful labeling properties are investigated through theoretical analysis and computational verification using a Python algorithm that checks the labeling conditions and generates visualizations of labeled graphs. The study provides a formal definition of the Upstairs graph and presents a general theorem stating that (Us_n) admits a graceful labeling for every integer (n\geq1). Illustrative cases demonstrate the labeling construction, while computational results are consistent with the theoretical results for the tested instances. Thus, the Upstairs graphs constitute a new family of grid-derived graphs with graceful labeling properties.

Keywords: graceful labeling; graph construction; graph labeling; grid graph; Python algorithm; Upstairs graph.

Upstairs 图的优美标号

摘要：由于目前尚未建立优美图的一般性判定方法，判断任意给定图是否存在优美标号仍是图论中的一个开放问题。本研究引入基于网格图分层构造的新图结构——Upstairs 图，记为 (Us_n)。研究采用理论分析与计算验证相结合的方法，考察其优美标号性质； 其中，Python 算法用于检验标号条件并生成标号图的可视化表示。本文给出了 Upstairs 图的形式化定义，并提出一项一般性定理：对于任意整数 (n\geq1)，(Us_n) 均存在优美标号。 示例展示了标号的构造方法；对于所测试的实例，计算结果与理论结果一致。因此，Upstairs 图构成一类由网格图衍生、具有优美标号性质的新图族。

关键词：优美标号；图构造；图标号；网格图；Python 算法；Upstairs 图。

1. Introduction

Graph theory has become one of the most active areas in discrete mathematics due to its broad theoretical development and practical applicability across various domains, including coding theory [1], cryptography [2], [3], and complex network modelling [4]. Among its rapidly growing branches, graph labeling has attracted substantial attention because of its ability to assign mathematical structures to graph elements through specific labeling rules. In general, a labeling of a graph $G = (V(G), E(G))$ is a mapping that assigns labels from a prescribed set, typically a set of non-negative integers, to the vertices, edges, or both of G, subject to specified conditions. Different labeling schemes are defined according to the constraints imposed on the vertex and/or edge labels and are used to characterize various structural properties of graphs [5].

One of the most extensively studied labeling schemes is graceful labeling, introduced by Rosa in 1967. A graph G is said to admit a graceful labeling if there exists an injective vertex-labeling function $\phi: V(G) \to \{0, 1, \ldots, | E(G) |\}$, such that the induced edge-labeling function $\phi^*: E(G) \to \{1, 2, \ldots, | E(G) |\}$ defined as $\phi^*(uv) = | \phi(u) - \phi(v) |$ is bijective for every edge $uv \in E(G)$ [6], [7]. A graph G is referred to as graceful if it admits a graceful labeling [8], [9]. Moreover, a graph G is called graceful if its vertices can be assigned distinct labels from a prescribed interval such that the absolute differences of labels assigned to adjacent vertices produce edge labels that form a complete and non-repeating sequence [10], [11]. Graceful labeling has become an important topic in graph theory because not every graph admits graceful labeling, and until now no complete characterization of graceful graphs has been established. Consequently, identifying new graph classes that satisfy graceful labeling remains an open and relevant research direction in contemporary graph theory.

Previous studies have investigated graceful labeling in numerous graph families, including path graphs [12], star graphs [13], [14], spider graphs [15], [16], tree-related graphs [17], ladder graphs [18], cycle-derived graphs [19], algebraic graphs [20]. Recent developments have demonstrated that introducing new graph structures through systematic graph transformations may lead to additional classes of graceful graphs [21]. In particular, studies on the W graph showed that combining fundamental graph structures (ladder graphs L_{n+1} and circle graph C_3) can generate new graceful constructions [22]. More recently, our previous work introduced the Vee graph as a new graph structure derived from two grid graphs $(P_2 \times P_n)$ and $(P_2 \times P_{n+1})$ attached at corresponding end edges, and it was proven to admit graceful labeling for all admissible graph orders [23]. The present study moves beyond this attachment scheme by considering a recursively generated layered construction, leading to a different graph family with distinct vertex and edge cardinalities, attachment mechanism, structural organization, and labeling formulas. This distinction motivates the introduction of the Upstairs graph (Us_n) and the investigation of its graceful labeling as a new problem rather than as a geometric modification of the Vee graph. In parallel, recent computational studies incorporated C++ and Python-based algorithms to verify graph labeling patterns and visualize labeling behaviour systematically [1], [18]. These findings indicate that several modified graph constructions provide a promising foundation for developing broader classes of graceful graphs and motivate further exploration of alternative construction patterns.

Although several modified grid-based graphs have been studied, existing works remain limited to relatively simple attachment mechanisms and do not sufficiently explore hierarchical or layered graph arrangements. To the best of our knowledge, no study has formally introduced or investigated a graph constructed through a stair-like arrangement of grid components together with its graceful labeling properties and computational approaches. This limitation opens an opportunity to extend the current state of the art by proposing a new graph model with a different structural configuration while preserving the combinatorial richness of grid graphs.

Motivated by the need to expand graceful labeling studies beyond conventional graph classes, this work introduces a new graph structure called the Upstairs graph, denoted by Us_n, constructed through a tiered stair-like arrangement of grid graph components. The proposed construction generates a distinctive adjacency configuration that differs from classical grid graphs and previously developed grid-derived structures. The graph is formally established through mathematical definitions of its structural properties, including the corresponding vertex and edge sets, with the objective of providing a new object for graceful labeling investigation and extending the existing family of grid-based graphs.

To analyze its labeling behavior, this study adopts a theoretical-analytical approach consisting of graph construction, structural characterization, formulation of vertex labeling, derivation of induced edge labeling, and computational verification of graceful conditions. The constructed labeling scheme ensures injectivity on vertices and bijectivity on induced edge labels, leading to a general proof of graceful labeling for every integer $n \geq 1$. The results confirm that the Upstairs graph belongs to the family of graceful graphs and contribute theoretically by proposing a new construction paradigm that may support future developments in generalized graph labeling [24], graph transformations, and studies involving uncertain or fuzzy graph models [25], [26].

2. Methods

This study employed a theoretical and computational approach in graph theory using Python-based algorithm to investigate the graceful labeling properties of a newly introduced graph structure called the Upstairs graph, denoted by Us_n. The research procedure was conducted through several systematic stages as follows.

2.1. Literature Review

The first stage involved an extensive review of literature related to graph theory, graph labeling, graceful labeling, and grid graphs. Previous studies concerning graceful labeling on various graph classes were also examined to establish the theoretical foundation and identify research gaps relevant to the graph construction.

2.2. Construction of the Upstairs Graph

A new graph structure, namely the Upstairs graph Us_n, was constructed based on grid graphs $P_2 \times P_3$ with a tiered construction pattern resembling an Upstairs. After constructing the graph, the vertex set and edge set of Us_n were formally defined. The cardinalities of vertices and edges were then determined as $|V(Us_n)|$ and $|E(Us_n)|$. These structural properties served as the basis for developing the graceful labeling scheme.

2.3. Definition of Vertex Labeling Function

A vertex labeling function

$$\phi: V(Us_n) \to \{0,1,2,\dots,q\},$$

where $q = |E(Us_n)|$, was defined recursively for all vertices of Us_n. The labeling formulas were established according to the parity of n. The construction of the labeling function aimed to ensure that each vertex received a distinct label within the prescribed labeling interval.

2.4. Proof of Injectivity of the Vertex Labeling

The injective property of the vertex labeling function ϕ was then proven analytically. This step was carried out by demonstrating that no two distinct vertices shared the same label value. Consequently, the labeling function satisfied the injectivity requirement of graceful labeling [27].

2.5. Mathematical Proof of the Graceful Property

Based on the vertex labeling function, an induced edge labeling function

$$\phi^*(uv) = |\psi(u) - \psi(v)|$$

was constructed for every edge $uv \in E(Us_n)$. An edge labeling is derived based on the absolute difference of labels between adjacent vertices [28].

2.6. Proof of Bijectivity of the Edge Labeling

The next stage involved proving that the induced edge labeling function

$$\phi^*: E(Us_n) \to \{1,2,\dots,q\}$$

is bijective. The proof was conducted by showing that every edge obtained a unique label (injective property), and all integers from 1 to q appeared exactly once as edge labels (surjective property). Therefore, the induced edge labeling function satisfied the bijectivity condition required for graceful labeling [29], [30].

2.7. Verification of Graceful Labeling Property

After establishing the injectivity of the vertex labeling and the bijectivity of the edge labeling, it was concluded that the graph Us_n satisfies the definition of graceful labeling [6], [7]. Hence, the Upstairs graph Us_n was proven to be a graceful graph for every integer $n \geq 1$. This stage constitutes the core of the research, namely the proof that the Upstairs graph has a graceful labeling. The proof is conducted deductively by demonstrating that the vertex labeling function is injective and the edge labeling function is bijective, and that they cover the entire required set of labels

[27], [29], [30]. The analysis is performed for Upstairs graphs of general n.

2.8. Illustration and Validation

To validate and clarify the theoretical results, this stage presents illustrative examples of graceful labeling on Upstairs graph with several variation of values n, specifically $n = 3$ and $n = 4$. These illustrations aim to facilitate understanding of the graph structure and labeling patterns, as well as to strengthen the arguments supporting the research findings. Moreover, given the Python-based algorithm to validate the research finding. Otherwise, these examples visually demonstrated the correctness and consistency of the labeling construction.

3. Results and Discussion

This section presents the introduction and construction of a new graph family called the Upstairs graph, denoted by Us_n together with its structural properties. We also demonstrate its graceful labeling characteristics through systematic graph construction and labeling analysis.

3.1. Construction of the Upstairs Graph

A new graph structure, namely the Upstairs graph Us_n, is constructed by extending the grid graph $P_2 \times P_3$ through a recursive layered arrangement that forms an upward Upstairs pattern. Starting from $P_2 \times P_3$ as the initial layer, each subsequent layer is attached to a designated edge of the previous layer while preserving graph connectivity and adjacency relationships. This iterative construction generates a hierarchical graph structure with increasing complexity until the graph reaches n layers. Formally, the Upstairs graph is defined as follows.

Definition 1: Let $n \geq 1$ be an integer. The Upstairs graph, denoted by Us_n, is a finite simple graph which is obtained from the grid graph $P_2 \times P_3$ as the initial layer, then additional layers are attached recursively to one of the edges defined as $b_{n-1}c_{n-1}$. Thus, n denotes the number of staircase layers, with $Us_1 = P_2 \times P_3$. Formally, the Upstairs graph Us_n is characterized by the vertex set $V(Us_n)$ and edge set $E(Us_n)$ respectively are

$$V(Us_n) = V_a \cup V_b \cup V_c \cup V_D$$
$$= \{a_i | i = 0,1,2,\dots,n-1\} \cup \{b_j | j = 0,1,2,\dots,n\}$$
$$\cup \{c_k | k = 0,1,2,\dots,n\} \cup \{d_l | l = 0,1,2,\dots,n-1\}$$

and

$$E(Us_n) = E_{ab} \cup E_{bc} \cup E_{cd}$$
$$= \{a_i b_j | \forall i = j \ dan \ j = i+1\}$$
$$\cup \{b_j c_k | \forall j = k \ dan \ j = k+1\}$$
$$\cup \{c_k d_l | \forall k = l \ dan \ k = l+1\}.$$

From the construction pattern, the four vertex classes have cardinalities $|V_a| = |V_d| = n$ and $|V_b| = |V_c| = n+1$. Therefore, the total numbers of vertices

$$|V(Us_n)| = n + (n+1) + (n+1) + n = 4n + 2.$$

Similarly, $|E_{ab}| = |E_{cd}| = 2n$ and $|E_{bc}| = 2n+1$. So that, the total numbers of edges

$$|E(Us_n)| = 2n + (2n+1) + 2n = 6n + 1.$$

Hence, Us_n has n staircase layers, graph order $|V(Us_n)| = 4n + 2$ and size $|E(Us_n)| = 6n + 1$. In particular, the initial graph Us_1 has 6 vertices and 7 edges, and each increment from Us_1 to Us_{n+1} adds exactly 4 vertices and 6 edges. Furthermore, Figure 1 displays a visual representation of the graph Us_n.

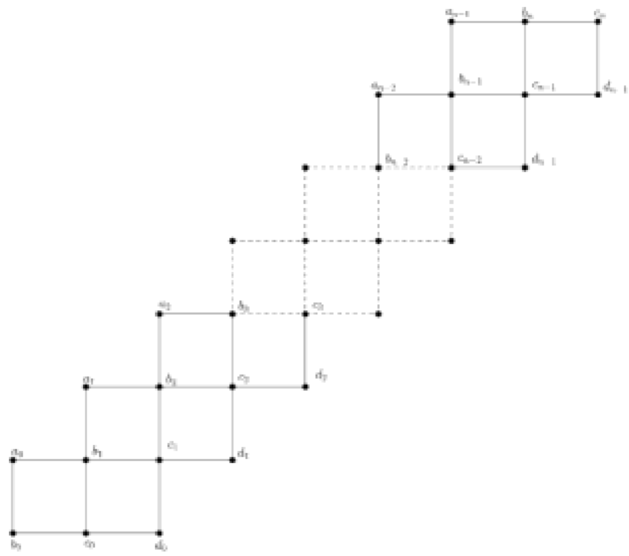

Figure 1. Upstairs graph Us_n

Now, we present example of Upstairs graph Us_n for $n = 2$ and $n = 3$.

Example 1: Figure 2 illustrated an example of Us_2 which contains from two layers grid graphs $P_2 \times P_3$ and attached in an edge b_1c_1,. This graph has 10 vertices and 13 edges which vertex set and edge set respectively

$V(Us_2) = \{a_0, a_1\} \cup \{b_0, b_1, b_2\} \cup \{c_0, c_1, c_2\} \cup \{d_0, d_1\}$

and

$E(Us_2) = \{a_0b_0, a_1b_1, a_0b_1, a_1b_2\} \cup \{b_0c_0, b_1c_1, b_2c_2, b_1c_0, b_2c_1\} \cup \{c_0d_0, c_1d_1, c_1d_0, c_2d_1\}.$

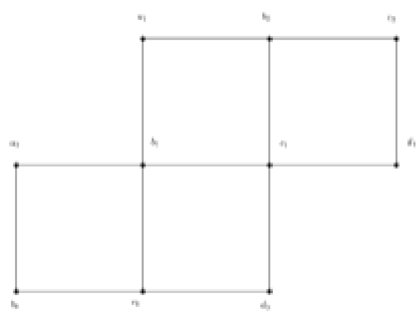

Figure 2. Upstairs Graph for $n = 2$ (Us_2)

Figure 3 illustrated an example of Us_3 which contains from three layers grid graphs $P_2 \times P_3$ and attached in edge b_1c_1 and b_2c_2. The Us_3 graph has 14 vertices and 19 edges which vertex set and edge set respectively

$$V(Us_3) = \{a_0, a_1, a_2\} \cup \{b_0, b_1, b_2, b_3\} \cup \{c_0, c_1, c_2, c_3\} \cup \{d_0, d_1, d_2\}$$

and

$$E(Us_3) = \{a_0b_0, a_1b_1, a_2b_2, a_0b_1, a_1b_2, a_2b_3\} \cup \{b_0c_0, b_1c_1, b_2c_2, b_3c_3, b_1c_0, b_2c_1, b_3c_2\} \cup \{c_0d_0, c_1d_1, c_2d_2, c_1d_0, c_2d_1, c_3d_2\}.$$

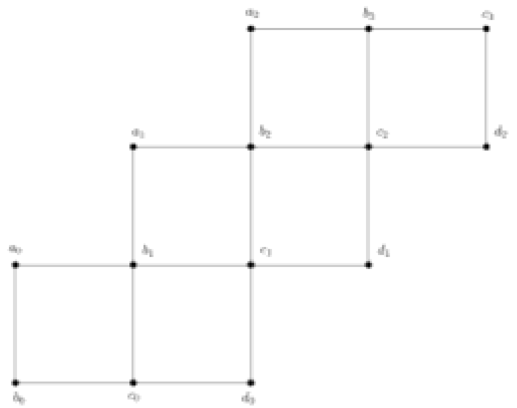

Figure 3. Upstairs Graph for $n = 3$ (Us_3)

3.2. Graceful Labeling of the Upstairs Graph

After introducing the construction of the Upstairs graph Us_n, we proceed to establish its graceful labeling property. The following theorem presents the main result of this study by demonstrating that the Upstairs graph admits a graceful labeling. By constructing an appropriate vertex labeling and examining the induced edge labeling, we prove that Us_n satisfies all conditions required for graceful labeling. The proof is carried out by defining an injective vertex labeling function and showing that the induced edge labeling generates a bijection over the required edge label set. The result is formally stated in the following theorem.

Theorem 1: For every integer $n \geq 1$, the Upstairs graph Us_n is graceful.

Proof: Define the labelling of the vertices on the graph Us_n as the function

$$\phi: V(Us_n) \to \{0,1,2, \dots, q\},$$

where $q = |E(Us_n)| = 6n + 1$.

$$\phi(a_i) = \begin{cases} 2 & ; \quad i = 0 \\ \phi(a_{i-1}) + 3; & i \neq 0 \end{cases} \tag{1}$$

$$\phi(b_j) = \begin{cases} q & ; \quad j = 0 \\ \phi(b_{j-1}) - 3; & j \neq 0 \end{cases} \tag{2}$$

$$\phi(c_k) = \begin{cases} 0 & ; \quad k = 0 \\ \phi(c_{k-1}) + 3; & k \neq 0 \end{cases} \tag{3}$$

$$\phi(d_l) = \begin{cases} q - 1 & ; \quad l = 0 \\ \phi(d_{l-1}) - 3; & l \neq 0 \end{cases} \tag{4}$$

For (1)-(4), the recursive formulas can be written explicitly as

$$\phi(a_i) = 3i + 2,$$
$$\phi(b_j) = q - 3j,$$
$$\phi(c_k) = 3k,$$
$$\phi(d_l) = q - 1 - 3l.$$

. For the index ranges in the construction of Us_n, these labels form the sets

$A = \{\phi(a_i); 0 \leq i \leq n\} = \{2,5,8, \dots, 3n + 2\}$,

$B = \{\phi(b_j); 0 \leq j \leq n\}$
$= \{q, q - 3, q - 6, \dots, 3n + 1\}$,

$C = \{\phi(c_k); 0 \leq k \leq n - 1\} = \{0,3,6, \dots, 3n - 3\}$,

$D = \{\phi(d_l); 0 \leq l \leq n - 1\}$
$= \{q - 1, q - 4, q - 7, \dots, 3n + 3\}$.

To prove that the vertex labeling function ϕ is injective, it is sufficient to show that distinct vertices receive distinct labels. Consider the labeling rules defined in Equation (1)-(4), the labels are generated recursively with increments or decrement of fixed magnitude 3 (depending on parity and special indices). Since each successive assignment modifies the previous value by a nonzero amount and follows a deterministic rule, no two distinct vertices $v_1 \neq v_2$ produce the same value. It remains to show that labels from different vertex classes cannot coincide.

The labeling scheme for vertices a_i, b_j, c_k, d_l (A, B, C, D) is also do not intersect, because they start from different initial values $2, q, 0, q - 1$ respectively and grow according to different arithmetic sequences. The labels A are all congruent to 2 (mod 3), whereas the labels in B are congruent to 1 (mod 3). Hence $A \cap B = \emptyset$. Similarly, $A \cap C = \emptyset$, $A \cap D = \emptyset$, $B \cap C = \emptyset$, $B \cap D = \emptyset$, because the corresponding residue classes module 3 are different. Finally, C and D are both multiples of 3, but $maxC = 3n - 3$ while $minD = 3n + 3$. Therefore, $C \cap D = \emptyset$. Thus, A, B, C, D are pairwise disjoint, implying that

$$\phi(v_1) \neq \phi(v_2), \forall\, v_1, v_2 \in V(Us_n).$$

Therefore, for any two vertices $v_1, v_2 \in V(Us_n)$,

$$\phi(v_1) = \phi(v_2) \Longrightarrow v_1 = v_2.$$

Hence, the vertex labeling function ϕ is injective. This implied that every vertex receives a unique label and is a subset of the set $\{0, 1, 2, \dots, q\}$ which is $\phi\left(V\left(Us(n)\right)\right) \subseteq \{0,1, \dots, q\}$.

Next, define the induced edge labeling function for the graph Us_n

$$\phi^*: E\,(Us_n) \to \{1, 2, \dots, q\}$$

where

$$\phi^*(uv) = |\phi(u) - \phi(v)|$$

for each edge $uv \in E(Us_n)$.

$$\phi^*(a_i b_j) = |\phi(a_i) - \phi(b_j)| = \begin{cases} |q-2| & ; \quad i=j=0 \\ |q-5| & ; \quad i=0, j=1 \\ |\phi(a_{i-1}) - \phi(b_{j-1}) + 6|; & otherwise \end{cases} \quad (5)$$

$$\phi^*(b_j c_k) = |\phi(b_j) - \phi(c_k)| = \begin{cases} |q| & ; \quad j=k=0 \\ |q-3| & ; \quad k=0, j=1 \\ |\phi(b_{j-1}) - \phi(c_{k-1}) - 6|; & otherwise \end{cases} \quad (6)$$

$$\phi^*(c_k d_l) = | \phi(c_k) - \phi(d_l) | = \begin{cases} |q-1| & ; \quad k=l=0 \\ |q-4| & ; \quad l=0, k=1 \\ |\phi(c_{j-1}) - \phi(d_{i-1}) + 6|; & otherwise \end{cases} \quad (7)$$

To complete the proof that the Upstairs graph Us_n admits a graceful labeling, it remains to show that the induced edge labeling function ϕ^* is bijective. From the vertex labeling defined in Equations (1)-(4), every vertex receives a distinct label. Consequently, each edge weight is generated from the absolute difference between two uniquely labeled endpoints. The induced edge labels are constructed according to Equations (5)-(7), covering every integer $n \geq 1$. Observe that the recursive construction of ϕ^* employs distinct arithmetic increments and decrements ± 6 across different edge category $a_i b_j, b_j c_k, c_k d_l$. Since each category edge produces values generated from different recursive offsets, no two distinct edges receive the same label. Therefore,

$$e_1 \neq e_2 \Longrightarrow \phi^*(e_1) \neq \phi^*(e_2)$$

for all $e_1, e_2 \in E(Us_n)$. Hence, $\phi^*$is injective.

Next, we show that every element of the codomain $\{1,2,\dots,q\}$ appears as an edge label. From Equations (5)-(7), the induced edge labels are generated systematically beginning from the largest values

$$q,\ q-1,\ q-2, \dots$$

and continuing through intermediate values until the smallest edge labels are obtained. According to the three edge classes of Us_n, the induced edge labels generated by (5)-(7) can be grouped into the following three arithmetic sequences:

$$S_{ab} = \{q - 2 - 3r; 0 \leq r \leq 2n-1\},$$
$$S_{bc} = \{q - 3r; 0 \leq r \leq 2n\},$$
$$S_{cd} = \{q - 1 - 3r; 0 \leq r \leq 2n-1\}.$$

Since $q = 6n+1$, these sets become

$$S_{ab} = \{6n-1, 6n-4, \dots, 2\},$$
$$S_{bc} = \{6n+1, 6n-2, \dots, 1\},$$
$$S_{cd} = \{6n, 6n-3, \dots, 3\}.$$

These three sets are mutually disjoint because they correspond to different residue classes modulo 3:

$S_{ab} \equiv 2\ (mod\ 3), S_{bc} \equiv 1\ (mod\ 3), S_{cd} \equiv 0\ (mod\ 3)$.

Therefore, $S_{ab} \cap S_{bc} = S_{ab} \cap S_{cd} = S_{bc} \cap S_{cd} = \emptyset$.

Moreover, every positive integer from 1 to $q = 6n+1$ belongs to exactly one of these three classes modulo 3. More explicity,

$$S_{ab} = \{2,5,8, \dots, 6n-1\},$$
$$S_{bc} = \{1,4,7, \dots, 6n+1\},$$
$$S_{cd} = \{3,6,9, \dots, 6n\}.$$

Hence,

$$S_{ab} \cup S_{bc} \cup S_{cd} = \{1,2, \dots, 6n+1\} = \{1,2, \dots, q\}.$$

Thus, the labeling formulas partition the edge set into mutually exclusive classes whose resulting differences collectively produce all integers in $\{1,2,\dots,q\}$. Thus,

$$Im(\phi^*) = \{1,2,\dots,q\}.$$

Therefore, for every integer $t \in \{1,2,\dots,q\}$, there exists an edge $e \in E(Us_n)$ such that $\phi^*(e) = t$. Hence, $\phi^*(E(Us_n)) = \{1,2,\dots,q\}$, which is ϕ^*surjective. Since the induced edge labeling function ϕ^* is both injective and surjective, it follows that ϕ^* is bijective.

Since the vertex labeling is injective and the induced edge labeling is bijective, according to the definition of graceful labeling, the labeling ϕ on Us_n is graceful, and consequently the Upstairs graph Us_n is a graceful graph for every integer $n \geq 1$. QED ■

3.3. Illustrating Cases of Graceful Labeling of the Upstairs Graph

To provide a clearer understanding of the Us_n graceful labeling scheme, illustrative examples of the Upstairs graph are presented corresponding to graph for $n = 3$ and $n = 4$. The following figures demonstrate the assignment of vertex labels and the resulting edge labels, showing that the induced labeling satisfies the graceful labeling criteria.

Example 2: Illustrations of graceful labeling on the Upstairs graph for $n = 3$ and $n = 4$ are shown in Figures 4 and 5, respectively.

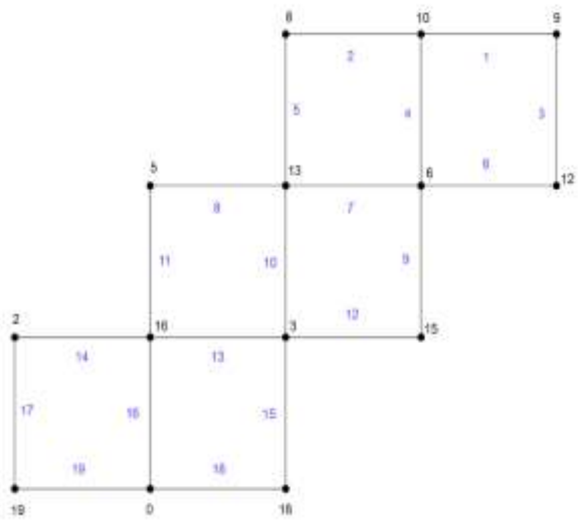

Figure 4. Graceful Labeling of Us_3

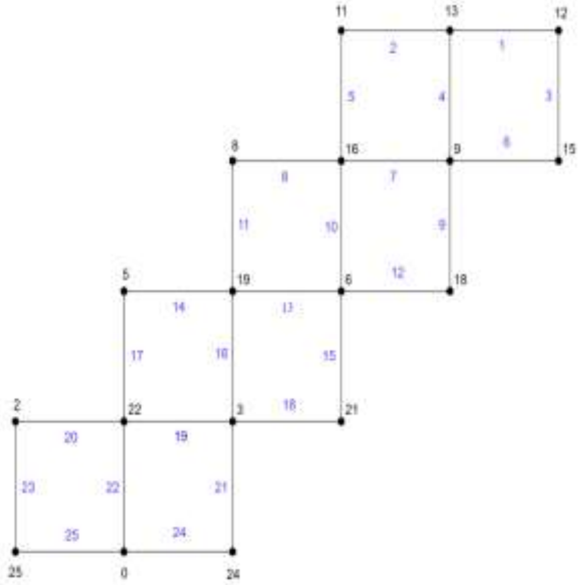

Figure 5. Graceful Labeling Us_4

3.4. Python-based Algorithm Implementation and Computational Verification of Graceful Labeling of the Upstairs Graph

To complement the theoretical proof of graceful labeling, a computational implementation was developed to generate and verify the labeling pattern of the Upstairs graph. Algorithm 1 was designed to automatically construct visualizations of graph structure for a given integer n, constructs the corresponding vertex and edge sets, assigns the vertex labels according to Equations (1)-(4), computes the induced edge labels, and checks whether the resulting edge label set is equal to $\{1,2,\dots,q\}$. Thus, the computational procedure serves as a reproducibility and validation for the labeling construction, whereas the validity of the graceful labeling for every integer $n \geq 1$ is established by the analytical proof of Theorem 1.

Algorithm 1 Graceful Labeling Construction for the Upstairs Graph Us_n

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Require: Integer $n \geq 1$
Ensure: Vertex labels, edge labels, and graceful verification

1: Construct the vertex set $V(Us_n)$.
2: Construct the edge set $E(Us_n)$.
3: Set $q = |E(Us_n)| = 6n + 1$.
4: Assign the vertex labeling function ϕ according to Equations (1)-(4).
5: Check that
$$\phi(v) \in \{0,1,\dots,q\}, \forall v \in \text{set } V(Us_n)$$
and that ϕ is injective.
6: **If** the conditions in Step 5 are satisfied, **then** continue; otherwise, **return** False.
7: **for all** $uv \in E(Us_n)$ compute
$$\phi^*(uv) = |\phi(u) - \phi(v)|.$$
8: **if** $\{\phi^*(uv) \mid uv \in E(Us_n)\} = \{1,2,\dots,q\}$ **then s**et verification result to True; otherwise, set it to False.
9: Visualize Us_n together with the vertex labels ϕ and induced edge labels ϕ^*.
10: **return** Verification result.

Algorithm 1 provides a compact computational procedure for implementing the general labeling construction and checking the graceful labeling conditions for a specified value of n. In particular, the algorithm does not replace the analytical proof of Theorem 1. It provides an independent computational confirmation that the proposed formulas are correctly implemented and that they produce the required labeling for the tested instance.

The computational experiments were performed in the Google Colab environment using Python version 3.13.15. NetworkX version 3.6.1 was used for graph construction and manipulation, while Matplotlib version 3.10.0 was used for graph visualization. The computational environment was kept consistent throughout the graph construction, vertex labeling, induced edge labeling, visualization, and verification procedures for the graceful labeling of Us_n.

For a specified value of n, the implementation constructs a graph with q edges, assigns the proposed vertex labels, computes the induced edge labels, and verifies whether the resulting edge-label set is exactly $\{1,2,\dots,q\}$. This computational procedure provides confirmation of the analytical construction for the selected instance and supports the reproducibility of the proposed labeling scheme. It should be emphasized that computational verification of individual or selected values of n does not by itself establish the result for all n. The general statement that Us_n is graceful for every integer $n \geq 1$ follows from the analytical proof given in Theorem 1.

Furthermore, the output visualization of the algorithm's implementation for the selected instance $n = 4$ is presented in Figure 6 to demonstrate consistency between the analytical derivation (Figure 5) and the computational results. The resulting vertex and induced edge labels are consistent with the labeling construction established analytically in Theorem 1, thereby providing computational confirmation of the proposed graceful labeling for this instance.

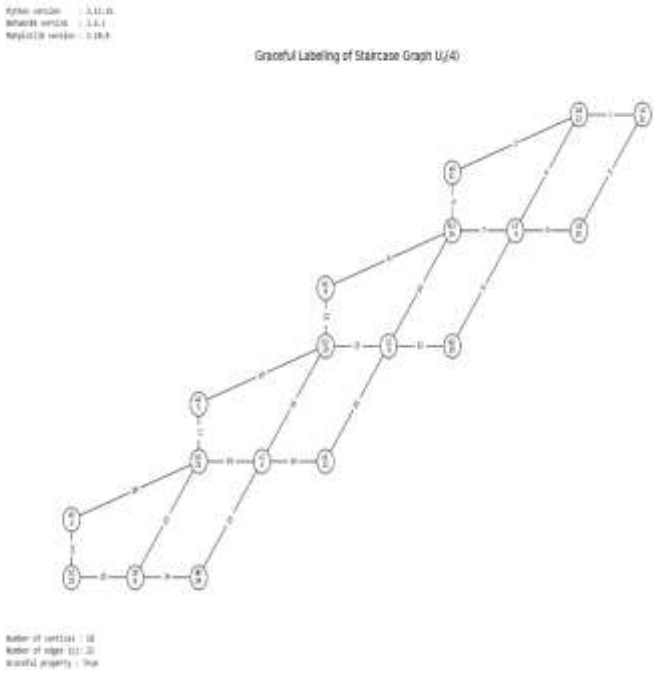

Figure 6. Output of Python-based Algorithm for Graceful Labeling Us_4

3.5. Distinction from the Vee Graph

Although the present study is motivated by our previous work on the Vee graph [23], the Upstairs graph (Us_n) is mathematically distinct from the Vee graph (V_n) in both its construction and labeling structure. The Vee graph (V_n) is obtained by attaching two grid graphs, ($P_2 \times P_n$) and ($P_2 \times P_{n+1}$), along one pair of corresponding end edges. Consequently, it has $4(n+1)$ vertices and $6n+4$ edges. In contrast, the Upstairs graph (Us_n) starts from the grid graph ($P_2 \times P_3$) and generates subsequent layers recursively by attaching each new layer to one of the edges defined as $b_{n-1}c_{n-1}$. This recursive mechanism produces $4n+2$ vertices and $6n+1$ edges, and gives the graph its characteristic Upstairs structure. Thus, the two graph families are not related merely by a geometric rearrangement of the same construction: the Vee graph is formed by a one-time attachment of two grids, whereas the Upstairs graph is defined through a recursive layer by layer construction.

The labeling structures are also fundamentally different. The graceful labeling of V_n uses two vertex classes (s_i and t_j) with several case dependent formulas involving the parity of n, special indices, and different increments or decrements. In the present construction, Us_n is described using four vertex classes (a_i, b_j, c_k, d_l) whose labels are generated from the initial values $2, q, 0, q-1$ respectively, through recursive arithmetic patterns with step size 3. The induced edge labeling is likewise organized according to three distinct edge classes (a_ib_j,b_jc_k, c_kd_l) and follows a different arithmetic difference pattern. Therefore, the graceful labeling established for Us_n is not a direct adaptation of the labeling scheme previously developed for V_n, but a new labeling construction associated with a different recursively generated graph family. These structural and labeling differences establish the mathematical novelty of the Upstairs graph and extend the class of grid-derived graphs known to admit graceful labeling.

3.6. Discussion

Previous studies primarily generated graceful graphs through structural composition. For example, the W graph was obtained through the combination of two ladder graphs and a cycle component [22] and the Vee graph was derived from two grid graphs [23], where graceful properties were established through explicit construction of vertex and induced edge labels. These approaches demonstrate that modifying known graph structures can produce additional graceful graph classes. In line, the findings of this study extend the current development of graceful graph construction by introducing a new graph family derived from a layered grid graph arrangement, namely Upstairs graph. This construction yields a structural configuration that differs from previously reported grid-derived graceful graphs and expands the diversity of known graceful graph families.

Another distinguishing contribution lies in the incorporation of computational verification. Recent studies have shown that C++ [18] and Python-based [1] algorithms can support the exploration and verification of graph labeling patterns and reveal structural regularities computationally. In this study, computational implementation is not intended as a substitute for proof, but serves as an additional validation mechanism to confirm the correctness of the theoretical labeling and to visualize graceful configurations of the Upstairs graph. The agreement between analytical proof and computational output strengthens the reproducibility of the proposed labeling framework.

Overall, these findings indicate that the Upstairs graph contributes not only as a new graceful graph family but also as an alternative construction paradigm for developing future grid-derived labeling structures and computational graph labeling studies.

4. Conclusion

In this paper, a novel graph construction called the Upstairs graph, denoted by Us_n, as a new recursively generated graph family and a distinct graceful labeling construction has been introduced and investigated within the framework of graceful labeling. The Us_n graph was developed from a stair-like arrangement of grid graph components, resulting in a structured yet distinctive topology that extends existing grid-derived graph families. The principal result is established analytically by providing a formal construction of the vertex labeling and proving that the induced edge labels satisfy the conditions of graceful labeling. Computational implementation is then employed as a complementary verification to validate the labeling construction and generate visual representations of the gracefully labeled graphs. Thus, the study establishes the Upstairs graph as a new class to the collection of known graceful graphs through a mathematically proven result supported by computational verification.

Future studies may extend the Upstairs graph through the development of structural variations and generalized stair-based graph families while examining their behavior under alternative graph labeling schemes, including edge graceful labeling, harmonious labeling, total labeling, and other generalized labeling frameworks. Beyond labeling theory, another promising direction is to investigate algebraic and spectral properties of the Upstairs graph through matrix-based representations and graph energy concepts. Inspired by recent developments in degree subtraction and degree square subtraction energies on structured grid-derived graphs [31], future studies may examine adjacency energy, Laplacian energy, signless Laplacian energy, Sombor energy, degree subtraction energy, and related spectral invariants of the Upstairs graph to identify hidden structural characteristics and asymptotic behavior. Moreover, recent advances in algebraic graph theory concerning chromatic polynomials of grid graphs have emphasized the importance of structured graph models in coloring and enumerative problem [32]. Given that the Upstairs graph originates from a grid-based construction, future work may examine whether similar algebraic or matrix-based approaches can be adapted to determine chromatic polynomials and other invariant properties of the Upstairs graph and its related variants.

Declarations

Author Contributions

Andika Ellena Saufika Hakim Maharani coordinated the research, formulated the Upstairs graph structure, constructed and proved the graceful labeling, and wrote the main manuscript.

Mamika Ujianita Romdhini contributed to the analysis of the graph structure, validation of graph properties, and development of the mathematical proofs.

Abdurahim contributed to the construction of labeling patterns, strengthening the generalization of the results, and identifying directions for further research and development.

Lingga Gita Dwikasari contributed to the preparation of graph illustrations, formal representation of graph structures, visualization, and interdisciplinary aspects of the research.

Natasya Dyahayu Lestari contributed to the literature review, preparation of supporting illustrations, and technical assistance throughout the research process.

All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The author declares that there is no conflict of interests regarding the publication of this manuscript.

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