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Fractional Calculus in the Solution of the Klein–Gordon Equation

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Abstract: This paper investigates the Klein–Gordon equation within the framework of fractional calculus by incorporating non-integer time and spatial derivatives to model physical processes characterized by memory effects and nonlocal interactions. Fractional operators in the Riemann–Liouville and Caputo senses are employed, together with the Laplace transform and the Mittag–Leffler function, to reformulate and solve the associated initial value problem. The resulting solutions are examined both analytically and graphically in order to evaluate the influence of fractional orders on the temporal and spatial dynamics of the system.

The analysis demonstrates that small variations in the fractional orders lead to significant qualitative changes in the behavior of the scalar field, indicating transitions between classical and fractional regimes. In particular, anomalous damping, power-law decay, and nonlocal propagation phenomena are observed, which are intrinsically linked to the properties of the Mittag–Leffler function.

The principal contribution of this study is the systematic characterization of the role of fractional order in the Klein–Gordon equation. The proposed fractional model generalizes the classical formulation and provides a suitable mathematical framework for describing systems exhibiting anomalous dissipation, long-term memory, and non-Euclidean geometric effects.

Keywords: Fractional calculus; Klein–Gordon equation; Caputo derivative; Riemann–Liouville derivative; Laplace transform; Mittag–Leffler function; nonlocal dynamics; memory effects.

分数阶微积分在克莱因–戈登方程求解中的应用

摘要：本文在分数阶微积分框架下研究克莱因–戈登方程，通过引入非整数阶时间与空间导数来刻画具有记忆效应和非局域相互作用的物理过程。采用 Riemann–Liouville 型与 Caputo 型分数阶算子，并结合拉普拉斯变换与 Mittag–Leffler 函数，对相关初值问题进行重构与求解。所得解通过解析与图形方法进行分析，以评估分数阶阶数对系统时空动力学行为的影响。

分析表明，分数阶阶数的微小变化会导致标量场行为发生显著的定性改变，体现出经典模型与分数阶模型之间的过渡特征。尤其表现为异常阻尼、幂律衰减以及非局域传播等现象，这些现象与 Mittag–Leffler 函数的性质密切相关。

本研究的主要贡献在于系统性刻画了分数阶阶数在克莱因–戈登方程中的作用。所提出的分数阶模型推广了经典形式，为描述具有异常耗散、长期记忆效应以及非欧几里得几何特征的系统提供了适用的数学框架。

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关键词：分数阶微积分；克莱因–戈登方程；Caputo 导数；Riemann–Liouville 导数；拉普拉斯变换；Mittag–Leffler 函数；非局域动力学；记忆效应

1. Introduction

The Klein–Gordon equation is a relativistic partial differential equation that describes the dynamics of scalar fields in theoretical physics. It is a relativistic generalization of the Schrödinger equation and is mainly used in the context of quantum mechanics and quantum field theory. The equation is named after the physicists Oskar Klein and Walter Gordon, who proposed it in 1926 [1]. To obtain a satisfactory numerical approximation with minimal error, we employ fractional calculus, which extends the notion of differentiation to non-integer orders [2]. Some advantages of seeking an approximate solution using fractional calculus include:

• **Anomalous diffusion:** Unlike classical diffusion, where particles spread uniformly, anomalous diffusion describes processes in which particles move non-uniformly, displaying complex dispersion patterns.

• **Memory effects:** Systems with memory retain information about their past states, which influences their present behavior.

• **Complex material behavior:** Certain materials exhibit physical properties that cannot be adequately described using classical models.

2. Fractional Calculus Operators

The foundation of fractional calculus lies in questioning whether integer-order derivatives could be extended to fractional orders while preserving mathematical coherence. The origin of this idea dates back to L'Hôpital on September 30, 1695 [3]. On that day, in a letter to Leibniz, L’Hôpital mentioned certain limitations in the way Leibniz’s convention for the nth derivative of a differentiable function was expressed. Driven by scientific curiosity, L'Hôpital asked what would happen if nnn were taken as one-half [4]. Leibniz responded with “an apparent paradox from which useful consequences will one day be drawn” [5].

This unprecedented exchange marked a turning point: soon after, brilliant minds such as Euler, Laplace, and Riemann, among others, turned their attention to fractional calculus, each contributing essential elements to its development. Historical records show that Lacroix was a precursor, publishing in 1819 the first work referring to fractional differential operators [6]. Starting with nnn, where nnn is a positive integer, Lacroix obtained the nth derivative, given by equation (1).

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n} \ \ with \ \ n \le m \qquad (1)$$

The Gamma function $\Gamma(x)$ is well defined and analytic for $x>0$ [7] [8]. Let us examine the usual results of the Gamma function. Note that the Gamma function satisfies, for all $x>0$ in its domain, the following property — see:

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x e^{-t} dt \\ &= \left.\frac{t^x}{e^t}\right|_0^\infty \\ &+ x\int_0^\infty t^{x-1} e^{-t} dt \\ &= x\Gamma(x)\end{aligned} \qquad (2)$$

Equation (2) expresses the property $\Gamma(x+1)=x\Gamma(x)$. This property allows us to derive a result for the Gamma function.

$$\Gamma(n+1) = n! \qquad (3)$$

We can use equation (3) and the Legendre symbo Γ, to write the n-ésima **Ошибка! Источник ссылки не найден.** derivative (1) of the function $f(x)=x^m$, we obtain equation (4).

$$\begin{aligned}\frac{d^n y}{dx^n} &= \frac{m!}{(m-n)!} x^{m-n} \\ \frac{d^n y}{dx^n} &= \frac{\Gamma(m+1)}{\Gamma(m-n+1)!} x^{m-n}\end{aligned} \qquad (4)$$

The Beta function is defined by equation (5):

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \quad with \quad x, y > 0 \qquad (5)$$

Applying the function $\Gamma(n)$ to the function $\beta(m,n)$. with $m>0$ a $n>0$, we obtain equation (6).

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad (6)$$

Although Lacroix was the first to publish on the concept, it was Abel who, in 1823, made the first effective use of fractional operators [6]. He applied differential operators of order $1/2$ to transform and solve the non-local integral equation characteristic of the tautochrone problem, thus establishing the first link between fractional calculus and classical mechanics.

One of the important results for fractional calculus is the Laplace transform of the derivative of order "n" of a function f. Si $f^{(n)}(t), n\in Z$ is piecewise continuous on $[0,\infty)$ and of exponential order, then the Laplace transform of its fractional derivative is given by equation (7).

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) \qquad (7)$$

Furthermore, the Laplace transform of the fractional integral of order "n" of a function f is given as follows. if $f:[0,\infty)\to R$ is piecewise continuous on $[0,\infty)$ and of exponential order, then the Laplace transform of its fractional integral is given by equation (8).

$$\mathcal{L}\{I^n f(t)\} = \frac{1}{s^n}\mathcal{L}\{f(t)\} = \frac{1}{s^n}F(s) \qquad (8)$$

Since the objective is to solve differential equations, when solving differential equations using the Laplace transform, what we obtain is the transform of our solution. It is therefore essential to recover the original function from its transform, which we achieve through the inverse Laplace transform, defined by equation (9).

$$\begin{aligned}f(t) &= \frac{1}{2\pi i}\lim_{R\to\infty}\int_{L_R} e^{st}F(s)\, ds \\ &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}F(s)\, ds \quad t>0\end{aligned} \qquad (9)$$

For a function of two variables $u(x,t)$ defined on $a\le x\le b$ and $t>0$, the Laplace transform with respect to t, is defined by equation (10), and its inverse transform is given by equation (11).

$$\begin{aligned}u(x,s) &= \mathcal{L}\{u(x,t)\} \\ &= \int_0^\infty e^{-st}u(x,t)\, dt\end{aligned} \qquad (10)$$

Its inverse transform is given by equation (11).

$$\begin{aligned}u(x,t) &= \mathcal{L}^{-1}\{u(x,s)\} \\ &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}u(x,s)\, ds \quad a>0\end{aligned} \qquad (11)$$

Let us note some relevant properties for the development of the text.

- $\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial x}\right\} = \frac{du(x,s)}{dx}$
- $\mathcal{L}\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} = \frac{d^2u(x,s)}{dx^2}$
- $\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial t}\right\} = su(x,s) - u(x,0)$
- $\mathcal{L}\left\{\frac{\partial^2 u(x,t)}{\partial t^2}\right\} = s^2u(x,s) - u(x,0) - \frac{\partial u(x,0)}{\partial t}$

These properties can be derived from the definition of the transform [9] for single-variable functions, using the method of integration by parts.

The Mittag–Leffler function is a generalization of the exponential function represented as a power series. Its application has proven particularly effective in the analysis of fractional differential equations, establishing itself as the natural analogue of the exponential function in the context of ordinary differential equations [10]. The Mittag–Leffler function is defined as follows in equation (12).

$$E_{\alpha,\beta} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{12}$$

When $\alpha=\beta=1$ we obtain: $E_{1,1}(z)=e^z$ which turns out to be just a particular case.

Fractional operators, such as the Riemann–Liouville and Caputo operators, each have their particular advantages depending on the type of problem to be solved. Note that the Caputo operator is suitable for initial value problems since its formulation allows initial conditions expressed in terms of continuous functions.

We will specify the equations for the integer-order derivative and the n-fold integrals. let f be a continuous and differentiable function on R . According to the classical definition, the expression for the derivative of order n.is given by equation (13).

$$\frac{d^n f(x)}{dx^n} = \lim_{h\to 0} \sum_{k=0}^{n} (-\mathbf{1})^{\boldsymbol{k}} \binom{\boldsymbol{n}}{\boldsymbol{k}} \frac{\boldsymbol{f(x+h)}}{h^n}; n \in \boldsymbol{N} \tag{13}$$

Where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$, we use the gamma notation and $n\in$N, it is necessary that $n\approx\alpha$ where $\alpha\in$Q using the properties of the gamma function we have equation (14).

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{h^\alpha} \lim_{h\to 0} \sum_{k=0}^{n} (-1)^k \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} f(x + kh) \tag{14}$$

Where nh is the maximum increment over the interval from x to b.

The derivative at the point x_0 exists, and the function f is continuous and differentiable throughout the entire interval I, Therefore, we can extend the derivative to the whole interval, which is, $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ For the minimum value of h as h→0 we get f' but the maximum value of h=b-x fo $x\in I(a,b)$ such that nh=b-x. Substituting into equation (14).

$$\frac{d^\alpha f(x)}{dx^\alpha} = D^\alpha f(x) = \lim_{h\to 0} \left(\frac{n}{b-x}\right)^\alpha \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f\left(x + \frac{k}{h}(b-x)\right) \tag{15}$$

Since h=$\frac{b-x}{n}$ and h→0 then $n\to\infty$, it follows that for D^α in equation (15) there exists α less than zero ($\alpha<0$) such that,

$$D^{-\alpha} f(x) = \lim_{h\to 0} \left(\frac{n}{b-x}\right)^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{-\alpha}{k} f\left(x + \frac{k}{h}(b -\right.$$
$$= \lim_{h\to 0} h^\alpha \sum_{k=0}^{n} (-1)^k \frac{(-\alpha)!}{k!(-\alpha-k)!} f(x+kh)$$

We introduce a representation for the combination of two numbers under some particular conditions of the equation (16), supported in the literature [11].

$$\binom{-\alpha}{k} = \frac{(-\alpha)!}{k!(-\alpha-k)!} = \frac{-\alpha(-\alpha-1)\ldots(-\alpha-k+1)(-\alpha-k)(-\alpha-k -}{k!(-\alpha-k)(-\alpha-k-1)..,1} = \frac{\overbrace{-\alpha(-\alpha-1)\ldots(-\alpha-k+1)}^{(k-\text{terms.})}}{k!} = \frac{(-1)^k[\alpha(\alpha+1)\ldots(\alpha+k-1)]}{k!}$$

It is defined that $\begin{bmatrix}\alpha\\k\end{bmatrix}=\frac{[\alpha(\alpha+1)\ldots(\alpha+k-1)]}{k!}$, we can rewrite equation (17).

$$\binom{-\alpha}{k} = (-1)^k \begin{bmatrix}\alpha\\k\end{bmatrix} \tag{18}$$

Thus, the result of equation (18) is substituted into equation (16).

$$D^{-\alpha} f(x) = \lim_{h\to 0} h^\alpha \sum_{k=0}^{n} \frac{(-\alpha)!}{k!(-\alpha-k)!} f(x+kh) = \lim_{h\to 0} h^\alpha \sum_{k=0}^{n} \begin{bmatrix}\alpha\\k\end{bmatrix} f(x+kh) \tag{19}$$

The derivative (19) is called the Grünwald-Letnikov derivative, used in the solution of systems.

Let's compare the writing of equation (19) for this: we can $f_h^{-\alpha}(x) = h^{\alpha}\sum_{k=0}^{n}\begin{bmatrix}\alpha\\k\end{bmatrix} f(x+kh)$ so we have equation (20)

$$D^{\alpha} f(x) = \lim_{h\to 0} f_h^{-\alpha}(x) \quad (20)$$

The Grünwald–Letnikov derivative is obtained in general terms as shown in equation (21).

$$ {}_xD_b^{-\alpha} = \frac{1}{(\alpha-1)!}\int_x^b (\tau - x)^{\alpha-1} f(\tau)\, d\tau \quad (21)$$

The foundations of fractional calculus begin with the consideration of the fractional integral. But before that, let us look at Cauchy's formula for the iterated integral, which we have used as a basis. Equation (22) allows us to simplify the integer-order n integral of a real function into a single convolution integral. We start by introducing the notation $I_{a^+}^{1}$ for the first integral of the function f that is, equation (22).

$$I_{a^+}^{1} f(x) = \int_a^x f(t)dt \quad (22)$$

Cauchy's formula for iterated integrals, considering the fundamental theorem of calculus and the method of integration by parts, can be written as equation (23).

$$I_{a^+}^{n} f(x) = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)dt \quad (23)$$

Cauchy's formula for iterated integrals precedes the Riemann–Liouville fractional integral of order α> 0, which naturally extends from this expression, as shown in equation (24).

$$I_{a^+}^{\infty} f(x) = \frac{1}{\Gamma(n)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad (24)$$

Let $f\in L^1[a,b]$ and $\alpha\in C$ be such that $R(\alpha)\geq 0$. The Riemann–Liouville integrals (24) of order a of the function f, denoted by $I_{a^+,t}^{\alpha}[f(t)]$ **Ошибка! Источник ссылки не найден.** and $I_{a^-,t}^{\alpha}[f(t)]$ **Ошибка! Источник ссылки не найден.**.

$$I_{a^+,t}^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, t \geq a \quad (25)$$

$$I_{b^-,t}^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)}{(x-t)^{1-\alpha}} dt, t < b \quad (26)$$

Called the right (25) and left (26) integrals, respectively. From now on, we will work with the right Riemann–Liouville fractional integral (25), since the left Riemann–Liouville fractional integral behaves similarly; for this reason, only the right integral will be used unless otherwise indicated.Let us note the linearity property.

$$I_a^{\alpha}\left(I_a^{\beta} f\right)(t) = \left(I_a^{\alpha+\beta} f\right)(t) \quad (27)$$

Equation (27) shows how the properties of the integral operator are satisfied.

If f is an absolutely continuous function on the interval $[a,b]$ [12] and $\alpha\in(0,1)$ then the Riemann–Liouville fractional integral of f of order $1-\alpha$ is also absolutely continuous on the same interval. Thus, the operator $I_a^{1-\alpha}$ is the Riemann–Liouville fractional integral of order $1-\alpha$, defined by equation (28).

$$(I_a^{1-\alpha} f)(t) = \frac{1}{\Gamma(2-\alpha)}\left(f(a)(t-a)^{1-\alpha} + \int_a^t f'(r)(t-r)^{1-\alpha} dr\right) \quad (28)$$

The integral property of a monomial is associated with an evaluation formula for the Riemann–Liouville fractional integral of functions of the form $(t\text{-}a)^{\beta}$, that is, a power of $t\text{-}a$, by substituting this expression into equation (25), we obtain.

$$I_a^{\alpha}\left[(t-a)^{\beta}\right] = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{(t-a)^{\beta}}{(t-r)^{1-\alpha}} dr \quad (29)$$

Notice that equation (29) can be rewritten in terms of the gamma function by employing the beta function (6). Consequently, the fractional integral of a monomial takes the form shown in equation (30).

$$I_a^{\alpha}\left[(t-a)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\beta+\alpha} \quad (30)$$

2.1. Derivada fraccionaria de Riemann Liouville

Let $f\in[a,b]\to R$, Then, the right Riemann–Liouville fractional derivative denoted by $\left(D_{a^+}^{\alpha} f\right)(t)$, Of order $\alpha>0$ is defined by equation (31).

$$\left(D_{a^+}^{\alpha}f\right)(t) = \frac{d^n}{dt^n}\frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}}ds, \quad t > a \tag{31}$$

Where $n=[\![\alpha]\!]+1$ and $[\![\alpha]\!]$ is the integer part of α.

Let $f\in[a,b]\rightarrow R$, the left Riemann–Liouville fractional derivative denoted by $(D_{b^-}^{\alpha}f)(t)$, Of order $\alpha>0$ it is defined as equation (32).

$$\left(D_{b^-}^{\alpha}f\right)(t) = \frac{d^n}{dt^n}\frac{1}{\Gamma(n-\alpha)}\int_t^b \frac{f(s)}{(t-s)^{\alpha-n+1}}ds, \quad t < b \tag{32}$$

Where $n = [\![\alpha]\!] + 1$.

2.2. Caputo derivative

Although the Riemann–Liouville fractional derivative has been fundamental in the theoretical development of non-integer order calculus, its formulation presents significant limitations in physical modeling. The main obstacle lies in the fact that the initial conditions associated with $(D^{\alpha-k}f)(0^+)$ with $\alpha\in(k\text{-}1,k]$ lack a direct physical interpretation, making their implementation in real systems difficult.

In contrast, the Caputo fractional derivative [13] uses initial conditions based on integer-order derivatives, that is, initial values that can be physically obtained in the traditional way. This definition has represented advances in the study and understanding of physical phenomena.

Let us define the Caputo fractional derivative of order α of f, where $f\in L_1(a,b)$ and $\alpha\in R^+$

$$D_a^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{\alpha}^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds \ \text{ for } t > a = I^{n-\alpha}\frac{d^n}{dt^n}f(t) \ \text{ for } t > a \tag{33}$$

The operator in equation (33) is subjected to the Laplace transform, as shown in equation (34).

$$\mathcal{L}\{D^{\alpha}f(t)\} = s^{\alpha}F(s) - \sum_{k=0}^{\infty} s^k f^{\alpha-k-1}(0) \tag{34}$$

If $l=\alpha\text{-}k\text{-}1$ then $f^{(l)}(0)$ is the value of the derivative of order l of the function f at the origin. Now, if we apply the Laplace transform to equation (33), it becomes:

$$\mathcal{L}\{I^{\alpha}f(t)\} = s^{-\alpha}F(s) \tag{35}$$

2.3. Fractional Differential Equations

In this introduction, we will explore the fundamental concepts, properties, and applications of fractional differential equations, emphasizing their importance in contemporary research and their potential to solve real-world problems [11].

Let the polynomial $P(x)=x^n+a_1x^{n\text{-}1}+a_2x^{n\text{-}2}+\cdots+a_n=0$, now we make $x=D^v$

$$P(D^v) = D^{vn} + a_1D^{v(n-1)} + \cdots + a_nD^0 = 0 \tag{36}$$

Equation (36) is known as the fractional differential operator, which can be written compactly as $P(D^v)y(t)=0$. We know that an ordinary differential equation (ODE) of order n has n linearly independent (L.I.) solutions.Now, a fractional differential equation must have an integer solution, that is, $N\geq nv$.

Theorem 2.
Let $P_n(D^v)y(t)=0$ be of order (n,q) and let the solution be $Y_1=L^{\text{-}1}\left(P^{\text{-}1}(s^v)\right)$. If N is the smallest integer such that $nv<N$, then $y_1,y_2,\ldots,y_i,\ldots,y_n$ are L.I solutions where $Y_{i+1}(t)=D^1\,Y_i(t)=0$ with $i=0,1,\ldots,N\text{-}1$

Theorem 3.
Let the equation $\left(D^{v(n)}+b_1D^{v(n\text{-}1)}+\cdots+b_nD^{v(o)}\right)y(t)=f(t)$ of order (n,q). Let the initial polynomial corresponding to the equation be given by $P(x)=x^n+a_1x^{n\text{-}1}+\cdots+a_n$. Let $y_1=L^{\text{-}1}\left(P^{\text{-}1}(s^v)\right)$ and let N be the smallest integer such that $N\geq nv$, Then $y_1,y_2,\ldots,y_i,\ldots,y_n(t)$ with $y_{j+1}(t)= D^jy_i(t)$ and $j=1,\ldots,N\text{-}1$. These are N L.I solutions.

3. Application of Fractional Calculus for the Solution of the Klein-Gordon Equation

The Klein-Gordon equation describes the dynamics of a scalar field $\varphi(X,t)$. It is a relativistic wave equation that includes terms such as mass and is derived from variational principles based on a Lagrangian density; that is, it is a function of spatial coordinates, velocities, and derivatives of the coordinates that describes how energy is distributed in a physical system throughout space and time [14] [15].

$$\left(\square+\frac{m^2c^2}{\hbar^2}\right)\varphi=0 \quad (37)$$

Where:

- $\varphi(X,t)$ is a scalar field
- c is the speed of light
- m the mass of the field
- $\hbar$ is the Planck constant
- D'Alambert Operator $\square=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2$
- With $\nabla^2=\sum_i \frac{\partial^2}{\partial x_i^2}=\Delta$ Laplacian operator.

The term $\square\varphi$ represents the propagation of the wave in space and time, so we can consider that it only moves along x, the term $m^2\varphi$ is the mass effect that modifies the wave propagation, which can be considered as φ and we will use natural units [16], that is, $\hbar=c=1$, to simplify equation (37), it can be written as equation (38).

$$\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}+m^2\varphi=0 \quad (38)$$

Equation (38), under certain initial conditions, leads to the initial value problem (39).

$$\begin{gathered}\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}+m^2\varphi=0\\ \varphi(x,0)=f(x)\\ \frac{\partial}{\partial t}\varphi(x,0)=g(x)\end{gathered} \quad (39)$$

The initial value problem (39) has different approximate solutions such as:

Classical solution

The solutions to the classical Klein-Gordon equation are plane waves of the form $\varphi(x,t)=e^{i(kx-\omega t)}$ where k is the wave vector and is the frequency. The dispersion relation $\omega^2=k^2+m^2$, describes how the different components of a wave propagate through space. This indicates that the solutions are waves propagating with a frequency ω related to the momentum k and the mass m [17]. The solutions can be real or complex, depending on the physical context.

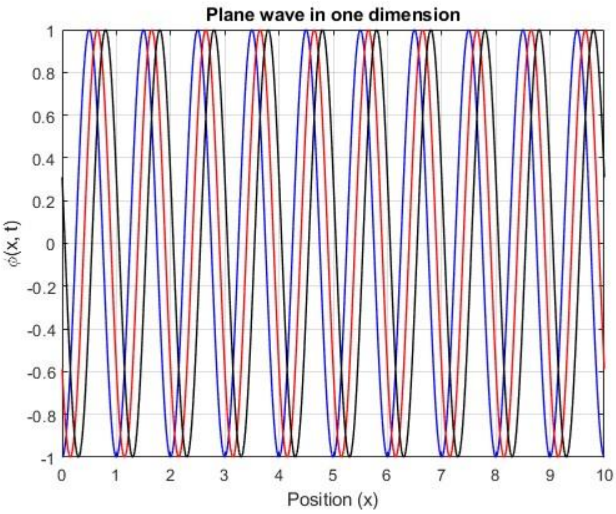


Figure 1. Solution of the Klein–Gordon equation for three values of .

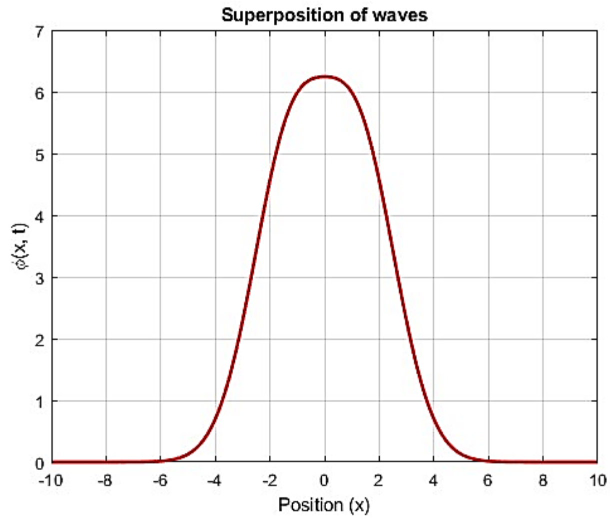


Figure 2. Different plane waves with different wave numbers k are superposed to form a localized wave packet.

3.1. Fractional Klein–Gordon equation.

In its traditional form, it uses second-order integer derivatives to model the temporal and spatial evolution of the particle. By applying the concept of fractional calculus, it will be extended to non-integer orders of differentiation. This means that we can consider derivatives of fractional order. By replacing the integer derivatives in the Klein-Gordon equation with fractional derivatives, we obtain the fractional Klein-Gordon equation in the spatial dimension x and temporal dimension t, which is given by equation (40).

$$\begin{gathered}\frac{\partial^\alpha\varphi}{\partial t^\alpha}-\frac{\partial^\beta\varphi}{\partial x^\beta}+m^2\varphi=0\\ \varphi(x,0)=f(x)\\ \frac{\partial}{\partial t}\varphi(x,0)=g(x)\end{gathered} \quad (40)$$

For $1<\alpha\leq 2$ and $1<\beta\leq 2$, where:

- $\varphi(x,t)$ is a scalar field.
- $\frac{\partial^\alpha \varphi}{\partial t^\alpha}$ It is the fractional derivative of φ with respect to time t of order α.
- $\frac{\partial^\beta \varphi}{\partial x^\beta}$ It is the fractional derivative of β with respect to space x of order β.
- m is the mass.

The form of the temporal derivative is given by equation (41).

$$D_0^\alpha \varphi(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\varphi^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau \quad (41)$$

And the form of the spatial derivative is given by equation (42).

$$D_0^\beta \varphi(x) = \frac{1}{\Gamma(n-\beta)}\int_0^x \frac{\varphi^{(n)}(s)}{(x-s)^{\beta+1-n}}ds \quad (42)$$

Continuing, we can assume that the function φ is separable, so, $\varphi(x,t)=X(x)T(t)$, then, for equations (41) and (42), we have:

- Temporal: $D_0^\alpha \varphi(x,t) = X(x)D_0^\alpha T(t) = X(x)\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{T^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau$ additionally $\mathcal{L}\{D_0^\alpha T(t)\} = s^\alpha \mathcal{L}\{T(t)\} - \sum_{k=0}^{\alpha-1} s^{\alpha-k-1}D^k T(0) = s^\alpha \mathcal{L}\{T(t)\} - s^{\alpha-1}T(0) - s^{\alpha-2}DT(0)$.
- Spatial: $D_0^\beta \varphi(x,t) = T(t)D_0^\beta X(x) = T(t)\frac{1}{\Gamma(n-\beta)}\int_0^x \frac{X^{(n)}(s)}{(x-s)^{\beta+1-n}}d\tau$ additionally $\mathcal{L}\left\{D_0^\beta X(x)\right\} = s^\beta \mathcal{L}\{X(x)\} - \sum_{k=0}^{\beta-1} s^{\beta-k-1}D^k X(0) = s^\beta \mathcal{L}\{X(x)\} - s^{\beta-1}X(0) - s^{\beta-2}DX(0)$.

hen, the initial value problem equation (40), considering that φ is separable, is:

$$X(x)D^\alpha T(t) - T(t)D_0^\beta X(x) + m^2 X(x)T(t) = 0 \quad (43)$$

Which equation (43) can be written as equation (44).

$$\frac{1}{T(t)}D^\alpha T(t) - \frac{1}{X(x)}D_0^\beta X(x) + m^2 = 0 \quad (44)$$

In the case that $m=0$ and $\frac{1}{T(t)}D^\alpha T(t) = \frac{1}{X(x)}D_0^\beta X(x) = \lambda^2$, we can find both temporal and spatial solutions.

3.1.1. Temporal solution

Equation (44), considering λ is written in the form:

$$\frac{1}{T(t)}D^\alpha T(t) - \lambda^2 + m^2 = 0 \quad (45)$$

Or equivalently, $D^\alpha T(t)+(m^2-\lambda^2)T(t)=0$. Solving this fractional differential equation using the Laplace transform, we have:

$$\mathrm{L}\left[D^\alpha T(t)+\left(m^2-\lambda^2\right)T(t)\right]=0$$
$$\mathrm{L}[D^\alpha T(t)]+\left(m^2-\lambda^2\right)\mathrm{L}[T(t)]=0$$
$$s^\alpha \mathrm{L}[T(t)]-s^{\alpha-1}T(0)-s^{\alpha-2}DT(0)+\left(m^2-\lambda^2\right)\mathrm{L}[T(t)]=0$$

Then

$$\mathrm{L}[T(t)]=\frac{s^{\alpha-1}T(0)+s^{\alpha-2}DT(0)}{s^\alpha+(m^2-\lambda^2)}$$
$$\mathrm{L}[T(t)]=T(0)\frac{s^{\alpha-1}}{s^\alpha+(m^2-\lambda^2)}+DT(0)\frac{s^{\alpha-2}}{s^\alpha+(m^2-\lambda^2)}$$
$$T(t)=T(0)\mathrm{L}^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha+(m^2-\lambda^2)}\right]+DT(0)\mathrm{L}^{-1}\left[\frac{s^{\alpha-2}}{s^\alpha+(m^2-\lambda^2)}\right]$$
$$T(t)=T(0)E_{(\alpha,1)}((m^2-\lambda^2)t^\alpha)+DT(0)\ t\,E_{(\alpha,2)}((m^2-\lambda^2)t^\alpha)$$

3.1.2. Temporal solution

Equation (44), considering λ is written in the form:

$$-\frac{1}{X(x)}D^\beta X(x) + \lambda^2 + m^2 = 0 \quad (46)$$

Or equivalently, $D^\beta X(x)+(\lambda^2+m^2)X(x)=0$. Solving this fractional differential equation using the Laplace transform, we have:

$$\mathrm{L}\left[D^\beta X(x)-(\lambda^2+m^2)X(x)\right]=0$$
$$s^\beta \mathrm{L}[X(x)]-s^{\beta-1}X(0)-s^{\beta-2}DX(0)-(\lambda^2+m^2)\mathrm{L}[X(x)]=0$$

Then

$$\mathrm{L}[X(x)]=\frac{s^{\beta-1}X(0)+s^{\beta-2}DX(0)}{s^\beta-(\lambda^2+m^2)}$$
$$\mathrm{L}[X(x)]=X(0)\frac{s^{\beta-1}}{s^\beta-(\lambda^2+m^2)}+DT(0)\frac{s^{\beta-2}}{s^\beta-(\lambda^2+m^2)}$$
$$X(x)=X(0)\mathrm{L}^{-1}\left[\frac{s^{\beta-1}}{s^\beta-(\lambda^2+m^2)}\right]+DX(0)\mathrm{L}^{-1}\left[\frac{s^{\beta-2}}{s^\beta-(\lambda^2+m^2)}\right]$$
$$X(x)=X(0)E_{(\beta,1)}(-(\lambda^2+m^2)t^\beta)+DX(0)\ \mathrm{t}\ E_{(\beta,2)}(-(\lambda^2+m^2)t^\beta)$$

4. Analysis and Results

The behavior of both variables will be shown through graphs, from their discretization to the solution in both the spatial and temporal parts, as well as the function $\varphi(x,t)$, all following the order of the fractional derivative. The discretization of the spatial variable follows a distribution given by the hyperbolic secant function, while the temporal variable is discretized in equal divisions of the interval.

Let's first consider the case where λ^2-m^2<0

4.1.1. Let's examine the spatial solutions.

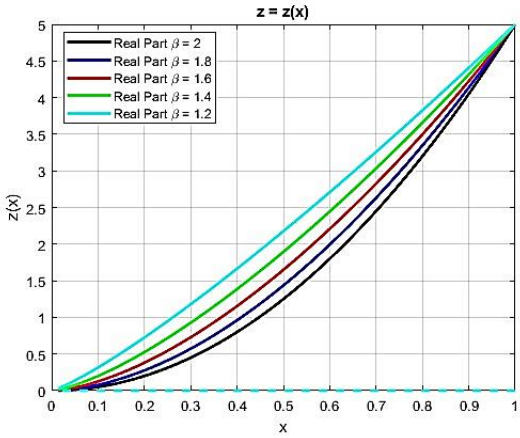


Figure 3. The function $z(x)=(\lambda^2+m^2)\,x^{\beta}$

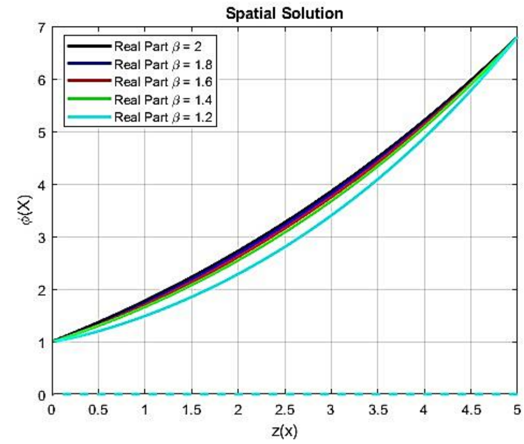


Figure 4. Spatial solution.

From Figure 3, it is observed that the function $z(x)=(\lambda^2+m^2)\,x^{\beta}$ g is a nonlinear potential that depends on x and grows monotonically from 0 up to (λ^2+m^2) since the term x^{β} is increasing. The parameter β is the fractional order, where for the case β=2, it corresponds to the classical case and the curve z is a parabola. Now, for β<2 the fractional case, the curve grows more slowly, showing a sub-quadratic behavior. It is evident that for all cases of β, the function $z(x)\to 0$ behaves similarly but with different slopes.

Now, Figure 4 shows the spatial solution of the Klein-Gordon equation. For β<2 , representing spatial smoothing, the solution $\varphi(X)$ shows fewer oscillations and greater decay due to the nonlocal characteristics of the Mittag-Leffler function. Due to memory effects, the long tails of the Mittag-Leffler functions introduce nonlocal spatial dependence, showing field propagation in media with spatial memory. A decay following a power law is observed, leading the function $\varphi(X)$ to exhibitanomalous damping. For the case β=2 , the classical solution X reduces to an exponential function $e^{z(x)}$.

4.1.2. Temporal solution

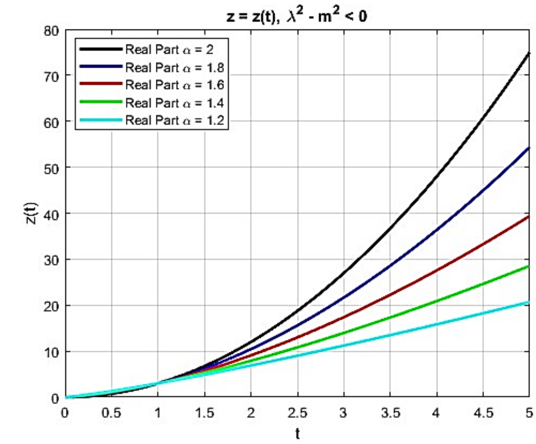


Figure 5. The function $z(t)$=-$(\lambda^2$-$m^2)\,t^{\alpha}$

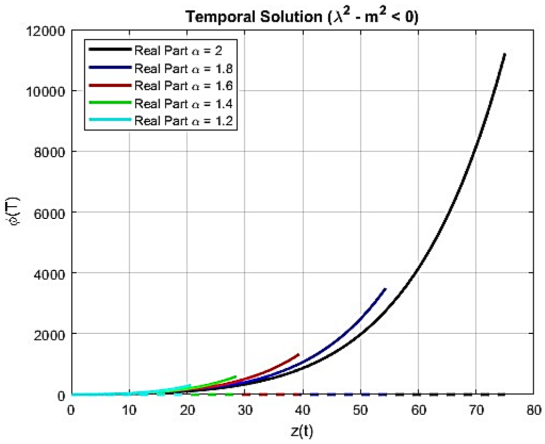


Figure 6. Temporal solution

From Figure 5, the function $z(t)$ is negative and decreases depending on α such that the scale is adjusted to $|\lambda^2$-$m^2|$.

Figure 6 shows the temporal solution of the Klein-Gordon equation. In the classical case, for α=2 the solution $\varphi(T)$, which is directly related to the Mittag-Leffler function, shows that the function T combines pure harmonic oscillations.

Now, in the fractional case, the function $\varphi(T)$ exhibits power-law decay with oscillations (referred to in English as power-law behavior), since it does not follow a simple exponential function but rather a power relation. The damped oscillations decay with an amplitude of t^{α}, and moreover, the nonlocality of the fractional derivative introduces temporal memory—that is, dependence on the history. The field $\varphi(T)$ oscillates with a frequency of $\omega=\sqrt{m^2\text{-}\lambda^2}$ but depends on α.

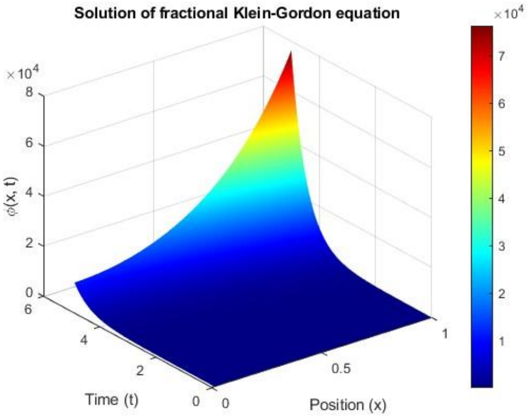


Figure 7. Classical solution for the case $\alpha=\beta=2$

Under these conditions, the fractional Klein-Gordon equation reduces to its classical version but with an effective imaginary mass term, which implies damped oscillatory behaviors or exponential growth.

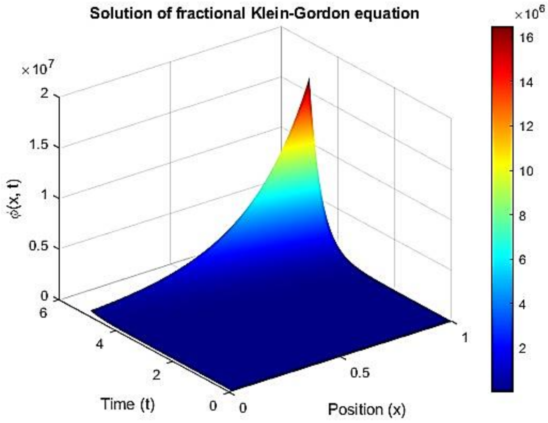


Figure 8. $\alpha = \beta = 1.2$

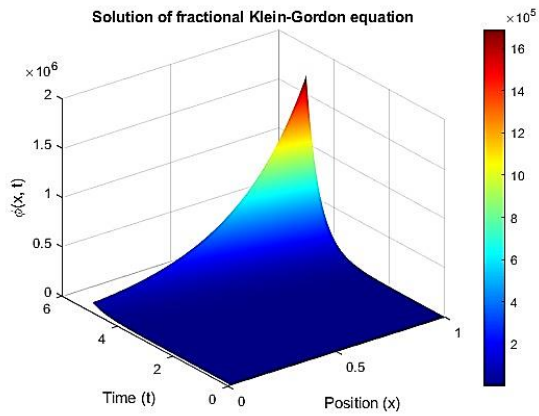


Figure 9. $\alpha = \beta = 1.4$

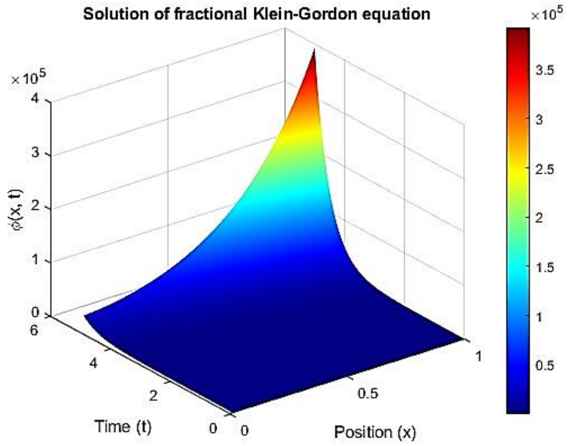


Figure 10. $\alpha = \beta = 1.6$

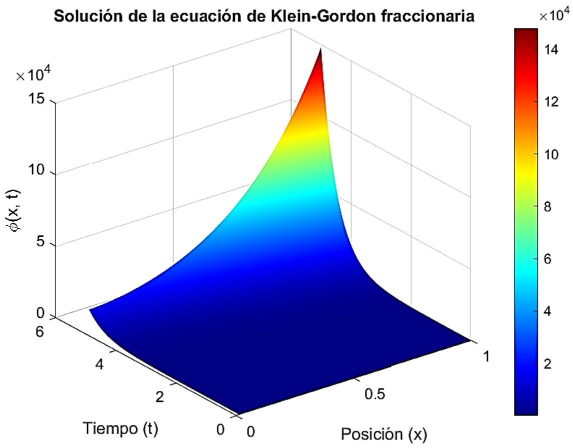


Figure 11. $\alpha = \beta = 1.8$

Figure 7 shows the general solution $\varphi(X,T)=X(x)T(t)$. The function φ combines both spatial and temporal components, where both depend on the Mittag-Leffler function.

Figure 8 shows the temporal part presenting a slow decay for $t^{1.2}$, while the sublinear growth of $x^{1.2}$, indicates media with high dissipation and propagated temporal memory.

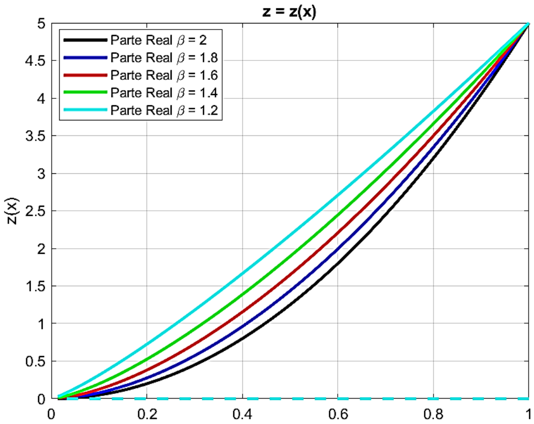
Figure 9 shows that the temporal decay is faster, yet still anomalous, and the spatial growth is sub-quadratic, which we expect between fractional and classical calculus.

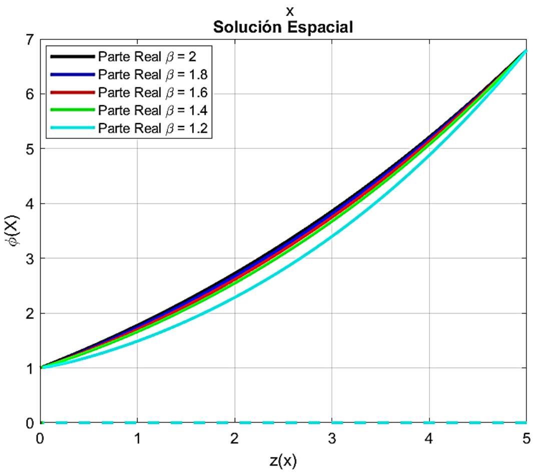
Figure 10 shows oscillations with moderate decay in the temporal part, and now the spatial part resembles quadratic growth, corresponding to a system with controlled dissipation and a quasi-classical spatial structure.

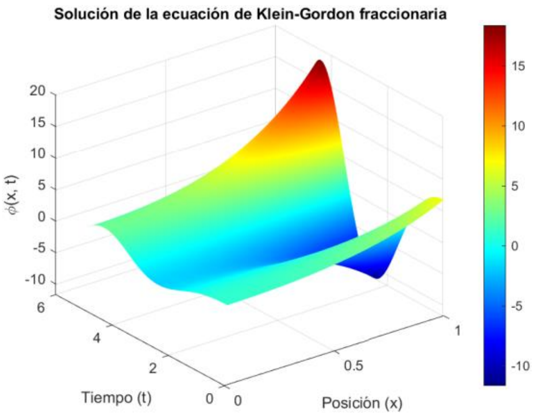
Figure 11 depicts the case near the limit before recovering classical behavior.

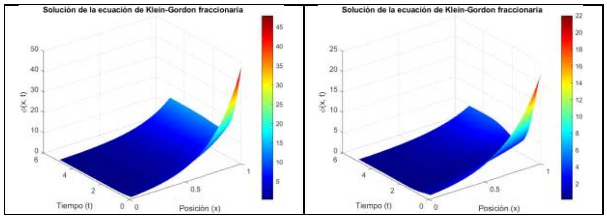
Let us first consider the case where λ^2-m^2>0

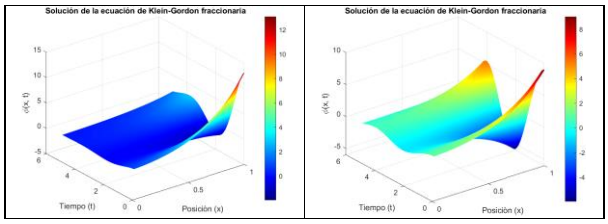
4.1.3. Let’s examine the spatial solutions.



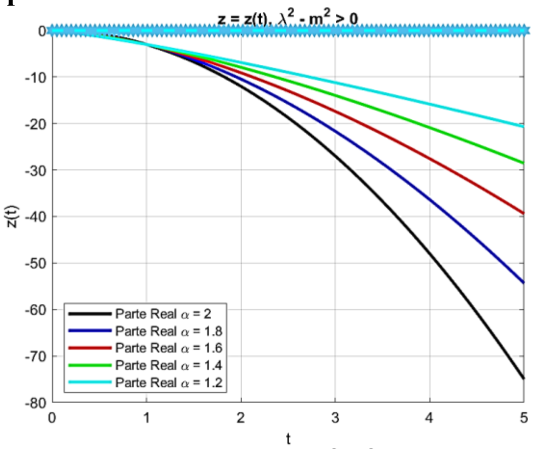


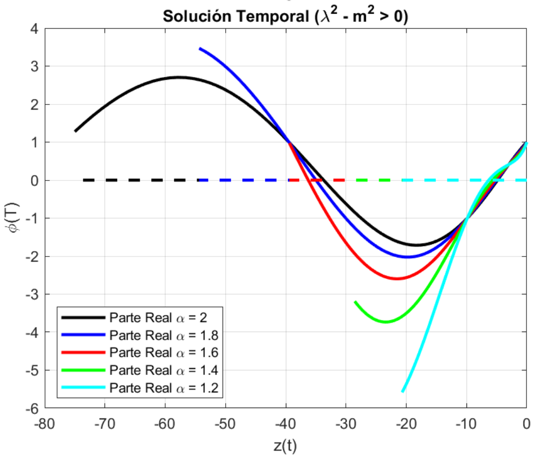






Temporal Solution





General Solution

5. Conclusions.

- This study demonstrates that the fractional formulation of the Klein–Gordon equation allows for the capture of non-classical effects associated with time memory and spatial non-locality. The solutions obtained show that fractional orders directly control the degree of damping, dispersion, and the transition between conservative and dissipative dynamics.
- In agreement with previous work on fractional differential equations and the use of the Mittag–Leffler function, the results confirm that the fractional regime generalizes classical exponential behavior. Unlike traditional integer models, the proposed formulation reproduces power-law decays and damped oscillations, which coincides with recent reports on physical systems with anomalous diffusion and complex media.
- From a theoretical perspective, the study reinforces the relevance of fractional calculus as a mathematical tool for extending fundamental equations of mathematical

physics. In applied terms, the results suggest that the fractional Klein–Gordon equation can be used to model materials with memory, quasi-elastic systems with minimal losses, and structures with fractal or non-Euclidean properties.

- The authors' main contribution lies in coherently integrating fractional operators with the Klein–Gordon equation, providing a detailed analysis of the spatial and temporal solutions and their dependence on fractional order. This approach expands our understanding of the physical impact of nonlocality and memory in relativistic models. Future research directions include studying energy conservation in the fractional framework, extending the model to higher dimensions, and incorporating nonlinear interaction terms. Furthermore, exploring numerical and experimental applications to compare the theoretical results with real-world physical systems is of interest.

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