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A General One-dimensional Tracer Transport Equation via the Dual Approach: Direct Integration of the Three-dimensional River Model

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Abstract: One-dimensional advection–dispersion formulations are widely used to model solute fate in rivers, yet they typically rely on restrictive cross-sectional averaging assumptions. Building on the dual approach, which formalizes dimensional reduction without ad hoc closures, we derive a general one-dimensional tracer transport equation by directly integrating the three-dimensional tracer conservation law for open-channel flow. The resulting formulation preserves the effects of cross-sectional heterogeneity through rigorously obtained correction terms that account for shear dispersion and transverse exchange, and it clarifies the conditions under which these terms vanish.

We show analytically that the previously proposed one-dimensional equation obtained by combining dual-approach reductions in the horizontal and vertical directions (the “two-horizontal/two-vertical” construction) is a special case of the present derivation. Under standard homogeneity and rapid-mixing limits, our equation reduces to the classical cross-sectional advection–dispersion model, ensuring backward compatibility with established practice. Validation against observed tracer data from a natural river reach demonstrates that the new formulation reproduces concentration evolution and longitudinal spreading more accurately than both the classical one-dimensional model and the combined two-horizontal/two-vertical dual-approach equation. By strengthening the theoretical foundations of dimension-reduced transport and improving empirical fidelity, the proposed equation offers a robust tool for analyzing pollutant dispersion and solute dynamics in fluvial environments and for guiding monitoring and management in rivers with pronounced lateral and vertical structure.

Keywords: classical cross-sectional averaging; dual approach; Huong River; one-dimensional tracer transport equation; three-dimensional tracer transport equation; advection–dispersion; shear dispersion.

基于对偶方法的一般一维示踪剂输运方程：三维河道流模型的直接积分

摘要

一维对流–弥散表述广泛用于模拟河流中溶质的迁移归宿，但通常依赖较为严格的横断面平均假设。基于对偶方法——该方法在无需经验性闭合的前提下形式化地实现降维——我们通过对明渠流中示踪剂守恒定律的三维方程进行直接积分，推导出一般性的一维示踪剂输运方程。所得方程通过严格推导的修正项保留了横断面非均质性的影响，这些修正项刻画了剪切弥散与横向交换，并明确了其可以忽略的条件。我们从理论上证明，此前通过在水平与垂向方向分别采用对偶方法降维并加以组合（即“二水平/二垂向”构造）得到的一维方程，是本文推导的一个特例。在标准的均匀性与快速混合极限下，所推导方程退化为经典的横断面对流–弥散模型，从而与既有实践保持向后兼容。以天然河段的实测示踪数据为基准的验证表明，相较于经典一维模型及上述“二水平/二垂向”对偶构造方程，新表述能更准确地再现浓度演化与纵向展宽。通过夯实降维输运的理论基础并提升经验吻合度，所提出方程为河流环境中的污染物扩散与溶质动力学分析提供了稳健工具，并可用于指导具有显著横向与垂向结构的河道的监测与管理。

关键词： 经典横断面平均；对偶方法；香江（Huong River）；一维示踪剂输运方程；三维示踪剂输运方程；对流–弥散；剪切弥散。

1. Introduction

The classical average equations or traditional average equation for flow and tracer transport have been introduced in many studies [1-6]. The dual conception involves considering two different aspects of a problem, which enables a more appropriate investigation. Dual averaging has been proved to be more general than classical averaging [7,8]. We applied dual averaging to develop all the equations for flow and tracer transport in one dimension (1D), two dimensional horizontal (2DH), two dimensional vertical (2DV), three dimensions (3D). This approach resulted in new equations and systems of equations that are more general than the classical ones [9-19]. Previously, to obtain the one-dimensional tracer transport equation in [11], we combined the two-dimensional horizontal from [9] and two-dimensional vertical from [11] tracer transport equations, both of

which were built using the dual approach.

In this paper, we develop a one-dimensional tracer transport equation directly using the dual approach from the classical three-dimensional tracer transport equation.

2. Building A One-dimensional Tracer Transport Equation

In this study, we derive a one-dimensional tracer transport equation directly from the classical three-dimensional tracer transport equation using the dual averaging approach. This method involves successive integration of the full 3D equation along the river's cross-section (y-direction) and vertical (z-direction) dimensions, capturing both local and global hydrodynamic characteristics.

First, we apply dual averaging along the y-axis to reduce the 3D equation to a two-dimensional vertical tracer transport equation. Next, we perform a double integration of this equation in the z-direction: (i) locally, from the riverbed to an intermediate depth, and (ii) globally, from the riverbed to the free surface.

In detail, we have the classical three-dimensional tracer transport equation as follows [1-6]:

$$\frac{\partial c}{\partial t}+\frac{\partial (uc)}{\partial x}+\frac{\partial (vc)}{\partial y}+\frac{\partial (wc)}{\partial z}=\frac{\partial}{\partial x}\left\{(D+\varepsilon_x)\frac{\partial c}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{(D+\varepsilon_y)\frac{\partial c}{\partial y}\right\}+\frac{\partial}{\partial z}\left\{(D+\varepsilon_z)\frac{\partial c}{\partial z}\right\} \quad (1)$$

The variables in the equation are defined as follows: x, y, z are the Cartesian coordinates; t represents time; *u, v, w* denote the velocity components along the *x, y, z* directions, respectively; D is the molecular diffusion coefficient; C stands for the conservative scalar variable related to tracer transport; and $\mathcal{E}_x$, $\mathcal{E}_y$, $\mathcal{E}_z$ are the turbulent diffusion coefficients in the x, y and z directions.

Using the dual averaging method for Equation (1) along the oy axis, the cross-sectional direction of the river, we obtained the two-dimensional vertical tracer transport equation [11] as follows:

$$\underbrace{\frac{1}{2}.\frac{\partial}{\partial t}\{\bar{C}(Y_2^2-Y_1^2)\}}_{(I)}-\underbrace{\alpha_{12}\bar{C}\left\{\frac{\partial}{\partial t}(Y_2^2-Y_1^2)\right\}}_{(J)}+\underbrace{\frac{1}{2}\beta_2\frac{\partial}{\partial x}\{\overline{UC}.(Y_2^2-Y_1^2)\}}_{(K)}-\underbrace{\beta_1\beta_3\overline{UC}\frac{\partial}{\partial x}(Y_2^2-Y_1^2)}_{(L)}+\underbrace{\frac{1}{2}\delta_2.\frac{\partial}{\partial z}\{\overline{WC}.(Y_2^2-Y_1^2)\}}_{(M)}-\underbrace{\delta_1\delta_3.\overline{WC}\frac{\partial}{\partial z}(Y_2^2-Y_1^2)}_{(N)}=\underbrace{\frac{1}{2}.\gamma_1.\gamma_2\frac{\partial}{\partial x}\left\{(D+\varepsilon_x).\frac{\partial\bar{C}}{\partial x}\right\}}_{(O)}.(Y_2^2-Y_1^2)+\underbrace{\frac{1}{2}.\theta_1.\theta_2\frac{\partial}{\partial z}\left\{(D+\varepsilon_z).\frac{\partial\bar{C}}{\partial z}\right\}.(Y_2^2-Y_1^2)}_{(P)} \quad (2)$$

where:

$\overline{C}$ represents the average concentration across the river width $b = Y_2 - Y_1$; $\overline{U}$, $\overline{W}$ denote the average velocity components along the river in the x and z directions, respectively; ε_x, ε_z are the turbulent diffusion coefficient in the x and z directions, respectively; D is the molecular diffusion coefficient of the tracer concentration; α_{ij}, β_i, δ_i, γ_i, θ_i are correction coefficients. Their values are close to 1 when the values C, UC, WC, $\frac{\partial c}{\partial x}$, $\frac{\partial c}{\partial z}$ are replaced with the average values $\bar{C}$, $\overline{UC}$, $\overline{WC}$, $\frac{\partial\bar{C}}{\partial x}$, $\frac{\partial\bar{C}}{\partial z}$ along the oy coordinate axis.

We continue to average Equation (2) in the Z direction using the dual approach, which means that: Integrating all terms of Equation (2) in the Z direction twice (Fig.1): first, through the local integration from Z_b to Z_m, and second, through the global integration from Z_b to Z_s. For simplicity, we ignore all overbars on the quantities in Equation (2).

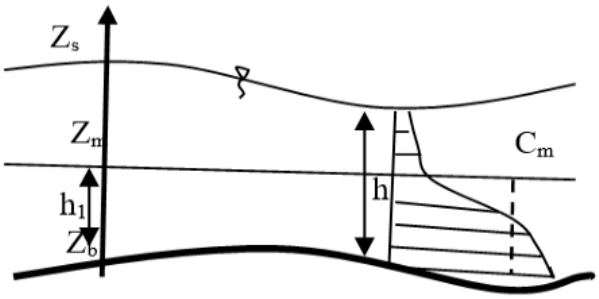

Fig.1. Integration sketch in the Z direction using the dual approach of the two-dimensional vertical tracer

The symbols in Fig.1 are as follows:
Z_b is the elevation of the riverbed;
Z_m is the elevation correspondent to any depth h_1, at this elevation, the concentration C_m is not significantly;
Z_s is the water surface elevation;
h is water depth.

Integrate the term (I) of Equation (2) for the first time:

$$TI=\frac{1}{2}\int_{Z_b}^{Z_m}\frac{\partial}{\partial t}\{C(Y_2^2-Y_1^2)\}dz=\frac{1}{2}\frac{\partial}{\partial t}\int_{Z_b}^{Z_m}\{C(Y_2^2-Y_1^2)\}dz-\frac{1}{2}\int_{Z_b}^{Z_m}C(Y_2^2-Y_1^2).\frac{\partial(dz)}{\partial t} \quad (3a)$$

$$TI=\frac{1}{2}\alpha_1\frac{\partial}{\partial t}\left\{\overline{C_1}(Y_2^2-Y_1^2)h_1\right\}-\frac{1}{2}\alpha_2\overline{C_1}(Y_2^2-Y_1^2).\frac{\partial h_1}{\partial t} \quad (3b)$$

where: α_1, α_2 are the correction coefficients, defined similarly to those in equation (2), calculated as follows:

$$\alpha_1=\frac{1}{\frac{\partial}{\partial t}\{\overline{C_1}(Y_2^2-Y_1^2)h_1\}}.\frac{\partial}{\partial t}\int_{Z_b}^{Z_m}\{C(Y_2^2-Y_1^2)\}dz \quad (3c)$$

$$\alpha_2=\frac{1}{\overline{C_1}(Y_2^2-Y_1^2).\frac{\partial h_1}{\partial t}}\int_{Z_b}^{Z_m}C(Y_2^2-Y_1^2).\frac{\partial(dz)}{\partial t} \quad (3d)$$

Calculate the integral of the term (I) of Equation (2) for the second time:

$$T^2I = \frac{1}{2}\underbrace{\alpha_1 \int_{Z_b}^{Z_s} \left\{\frac{\partial}{\partial t}\left[\overline{C_1}(Y_2^2 - Y_1^2)h_1\right]\right\} dh_1}_{I_1} - \underbrace{\frac{1}{2}\alpha_2 \int_{Z_b}^{Z_s} \overline{C_1}.(Y_2^2 - Y_1^2)\frac{\partial h_1}{\partial t} dh_1}_{I_2} \quad (3e)$$

$$I_1 = \frac{1}{2}\alpha_1 \int_{Z}^{Z_s} \left\{\frac{\partial}{\partial t}\left[\overline{C_1}(Y_2^2 - Y_1^2)h_1\right]\right\} dh_1 = \underbrace{\frac{1}{2}\alpha_1 \frac{\partial}{\partial t}\int_{Z_b}^{Z_s} \overline{C_1}(Y_2^2 - Y_1^2)h_1 dh_1}_{I_{11}} - \underbrace{\frac{1}{2}\alpha_1 \int_{Z_b}^{Z_s} \overline{C_1}(Y_2^2 - Y_1^2)h_1 \frac{\partial(dh_1)}{\partial t}}_{I_{12}} \quad (3f)$$

$$I_{11} = \frac{1}{2}\frac{\partial}{\partial t}\alpha_1 \int_{Z_b}^{Z_s} \overline{C_1}(Y_2^2 - Y_1^2)h_1 dh_1 = \frac{1}{4}\alpha_1\alpha_3.\frac{\partial}{\partial t}\{\overline{C}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\}; \quad (3g)$$

$$I_{12} = \frac{1}{2}\alpha_1 \int_{Z_b}^{Z_s} \overline{C_1}(Y_2^2 - Y_1^2)h_1 \frac{\partial(dh_1)}{\partial t} = \frac{1}{4}\alpha_1\alpha_4\overline{C}(Y_2^2 - Y_1^2).\frac{\partial}{\partial t}(Z_s^2 - Z_b^2) \quad (3h)$$

where:

α_3, α_4 are the correction coefficients, defined similarly to those in equation (2), calculated as follows:

$$\alpha_3 = \frac{2}{\frac{\partial}{\partial t}\{\overline{C}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2\}}\frac{\partial}{\partial t}\int_{Z_b}^{Z_s} \overline{C_1}(Y_2^2 - Y_1^2)h_1 dh_1\,; \quad (3i)$$

$$\alpha_4 = \frac{2}{\overline{C}(Y_2^2 - Y_1^2).\frac{\partial}{\partial t}(Z_s^2 - Z_b^2)}\int_{Z_b}^{Z_s} \overline{C_1}(Y_2^2 - Y_1^2)h_1 \frac{\partial(dh_1)}{\partial t}. \quad (3j)$$

On the other hand, we have:

$$I_2 \approx I_{12} \quad (3k)$$

Therefore:

$$T^2I = I_1 - I_2 = (I_{11} - I_{12}) - I_2 = I_{11} - 2I_{12} \quad (3l)$$

$$T^2I = \frac{1}{4}\alpha_1\alpha_3.\frac{\partial}{\partial t}\{\overline{C}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2\} - \frac{1}{2}\alpha_1\alpha_4\overline{C}(Y_2^2 - Y_1^2).\frac{\partial}{\partial t}(Z_s^2 - Z_b^2) \quad (3m)$$

Calculate the integral of the term (J) of Equation (2) for the first time:

$$TJ = -\alpha_{12}\int_{Z_b}^{Z_m} C\left\{\frac{\partial}{\partial t}(Y_2^2 - Y_1^2)\right\} dz = -\alpha_{11}\alpha_{12}\overline{C_1}.\frac{\partial}{\partial t}(Y_2^2 - Y_1^2).h_1 \quad (4a)$$

where:

α_{11} is the correction coefficient, defined similarly to that in equation (2), calculated as follows:

$$\alpha_{11} = \frac{1}{\overline{C_1}.\frac{\partial}{\partial t}(Y_2^2 - Y_1^2).h_1}\int_{Z_b}^{Z_m} C\left\{\frac{\partial}{\partial t}(Y_2^2 - Y_1^2)\right\} dz \quad (4b)$$

Calculate the integral of the term (J) of Equation (2) for the second time:

$$T^2J = -\alpha_{11}\alpha_{12}\int_{Z_b}^{Z_s} \overline{C_1}.\frac{\partial}{\partial t}(Y_2^2 - Y_1^2).h_1 dz = -\frac{1}{2}\alpha_{11}\alpha_{12}\alpha_{13}\overline{C}.\frac{\partial}{\partial t}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2) \quad (4c)$$

where:

α_{13} is the correction coefficient, defined similarly to that in equation (2), calculated as follows:

$$\alpha_{13} = \frac{2}{\overline{C}.\frac{\partial}{\partial t}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)}\int_{Z_b}^{Zs} \overline{C_1}.\frac{\partial}{\partial t}(Y_2^2 - Y_1^2).h_1 dz \quad (4d)$$

Calculate the integral of the term (K) of Equation (2) for the first time:

$$TK = \frac{1}{2}\beta_2 \int_{Z_b}^{Z_m} \frac{\partial}{\partial x}\{UC.(Y_2^2 - Y_1^2)\}dz \quad (5a)$$

$$TK = \underbrace{\frac{1}{2}\beta_2 \frac{\partial}{\partial x}\int_{Z_b}^{Z_m} UC.(Y_2^2 - Y_1^2)dz}_{TK_1} - \underbrace{\frac{1}{2}\beta_2 \int_{Z_b}^{Z_m} UC.(Y_2^2 - Y_1^2)\frac{\partial}{\partial x}(dz)}_{TK_2} \quad (5b)$$

$$TK = \underbrace{\frac{1}{2}\beta_1\beta_2 \frac{\partial}{\partial x}\left\{\overline{U_1C_1}.(Y_2^2 - Y_1^2).h_1\right\}}_{TK_1} - \underbrace{\frac{1}{2}\beta_2\beta_3\overline{U_1C_1}.(Y_2^2 - Y_1^2)\frac{\partial}{\partial x}(h_1)}_{TK_2} \quad (5c)$$

where:

β_1, β_3 are the correction coefficients, defined similarly to those in equation (2), calculated as follows:

$$\beta_1 = \frac{1}{\frac{\partial}{\partial x}\left\{\overline{U_1C_1}.(Y_2^2 - Y_1^2).h_1\right\}}\frac{\partial}{\partial x}\int_{Z_b}^{Z_m} UC.(Y_2^2 - Y_1^2)dz \quad (5d)$$

$$\beta_3 = \frac{1}{\overline{U_1C_1}.(Y_2^2 - Y_1^2)\frac{\partial}{\partial x}(h_1)}\int_{Z_b}^{Z_m} UC.(Y_2^2 - Y_1^2)\frac{\partial}{\partial x}(dz) \quad (5e)$$

Calculate the integral of the term (K) of Equation (2) for the second time:

$$T^2K = \underbrace{\frac{1}{2}\beta_1\beta_2 \int_{Z_b}^{Z_s} \frac{\partial}{\partial x}\left\{\overline{U_1C_1}.(Y_2^2 - Y_1^2).h_1\right\}dz}_{T^2K_1} - \underbrace{\frac{1}{2}\beta_2\beta_3 \int_{Z_b}^{Z_s} \overline{U_1C_1}.(Y_2^2 - Y_1^2)\frac{\partial}{\partial x}(h_1)dz}_{T^2K_2} \quad (5f)$$

Calculate the term T^2K_1:

$$T^2K_1 = \frac{1}{2}\beta_1\beta_2 \int_{Z_b}^{Z_s} \frac{\partial}{\partial x}\left\{\overline{U_1C_1}.(Y_2^2 - Y_1^2).h_1\right\}dz$$

$$T^2K_1 = \underbrace{\frac{1}{2}\beta_1\beta_2.\frac{\partial}{\partial x}\int_{Z_b}^{Z_s} \overline{U_1C_1}.(Y_2^2 - Y_1^2).h_1 dz}_{T^2K_{11}} - \underbrace{\frac{1}{2}\beta_1\beta_2 \int_{Z_b}^{Z_s} \overline{U_1C_1}.(Y_2^2 - Y_1^2).h_1.\frac{\partial(dz)}{\partial x}}_{T^2K_{12}} \quad (5g)$$

$$T^2K_1 = \underbrace{\frac{1}{4}\beta_1\beta_2\beta_4.\frac{\partial}{\partial x}\left\{\overline{UC}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\right\}}_{T^2K_{11}} - \underbrace{\frac{1}{4}\beta_1\beta_2\beta_5.\overline{UC}.(Y_2^2 - Y_1^2)\frac{\partial}{\partial x}(Z_s^2 - Z_b^2)}_{T^2K_{12}} \quad (5h)$$

where:

β_4, β_5 are the correction coefficients, defined similarly to those in equation (2), calculated as follows:

$$\beta_4 = \frac{2}{\frac{\partial}{\partial x}\{\overline{UC}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\}}.\frac{\partial}{\partial x}\int_{Z_b}^{Z_s} \overline{U_1C_1}.(Y_2^2 - Y_1^2).h_1 dz \quad (5i)$$

$$\beta_5 = \frac{2}{\overline{UC}.(Y_2^2 - Y_1^2)\frac{\partial}{\partial x}(Z_s^2 - Z_b^2)}\int_{Z_b}^{Z_s} \overline{U_1C_1}.(Y_2^2 - Y_1^2).h_1.\frac{\partial(dz)}{\partial x} \quad (5j)$$

Calculate the term T^2K_2:

$$T^2K_2 = -\frac{1}{2}\beta_2\beta_3\int_{Z_b}^{Z_s} \overline{U_1C_1}.(Y_2^2 - Y_1^2)\frac{\partial}{\partial x}(h_1)dz = -\frac{1}{4}\beta_2\beta_3\beta_6.\overline{UC}.(Y_2^2 - Y_1^2).\frac{\partial}{\partial x}(Z_s^2 - Z_b^2) \quad (5k)$$

where:

β_6 is the correction coefficient, defined similarly to that in equation (2), calculated as follows:

$$\beta_6 = \frac{2}{\overline{UC}.(Y_2^2 - Y_1^2).\frac{\partial}{\partial x}(Z_s^2 - Z_b^2)}\int_{Z_b}^{Z_s} \overline{U_1C_1}.(Y_2^2 - Y_1^2)\frac{\partial}{\partial x}(h_1)dz \quad (5l)$$

We notice:

$$T^2K_2 = T^2K_{12} \quad (5m)$$

Therefore, we have:

$$T^2K = T^2K_1 - T^2K_2 = (T^2K_{11} - T^2K_{12}) - T^2K_2 = T^2K_{11} - 2T^2K_2 \quad (5n)$$

$$T^2K = \frac{1}{4}\beta_1\beta_2\beta_4.\frac{\partial}{\partial x}\{\overline{UC}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\} - \frac{1}{2}\beta_2\beta_3\beta_6.\overline{UC}.(Y_2^2 - Y_1^2).\frac{\partial}{\partial x}(Z_s^2 - Z_b^2) \quad (5o)$$

Calculate the integral of the term (L) of Equation (2) for the first time:

$$TL = -\beta_1\beta_3\int_{Z_b}^{Z_m} UC\frac{\partial}{\partial x}(Y_2^2 - Y_1^2)dz = -\beta_1\beta_3\delta_1\overline{U_1C_1}\frac{\partial}{\partial x}(Y_2^2 - Y_1^2).h_1 \quad (6a)$$

where:

δ_1 is the correction coefficient, defined similarly to that in equation (2), calculated as follows:

$$\delta_1 = \frac{1}{\overline{U_1C_1}\frac{\partial}{\partial x}(Y_2^2 - Y_1^2).h_1}\int_{Z_b}^{Z_m} UC\frac{\partial}{\partial x}(Y_2^2 - Y_1^2)dz \quad (6b)$$

Calculate the integral of the term (L) of Equation (2) for the second time:

$$T^2L = -\beta_1\beta_3\delta_1\int_{Z_b}^{Z_s}\overline{U_1C_1}\frac{\partial}{\partial x}(Y_2^2 - Y_1^2)h_1dz = -\frac{1}{2}\beta_1\beta_3\delta_1\delta_2\overline{UC}.(Z_s^2 - Z_b^2).\frac{\partial}{\partial x}(Y_2^2 - Y_1^2). \quad (6c)$$

where:

δ_2 is the correction coefficient, defined similarly to that in equation (2), calculated as:

$$\delta_2 = \frac{2}{\overline{UC}.(Z_s^2 - Z_b^2).\frac{\partial}{\partial x}(Y_2^2 - Y_1^2)}\int_{Z_b}^{Z_s}\overline{U_1C_1}\frac{\partial}{\partial x}(Y_2^2 - Y_1^2)h_1dz \quad (6d)$$

For a one-dimensional problem, the velocity W in the Z direction is negligible, so:

$$T^2M = T^2N \simeq 0 \quad (7)$$

Calculate the integral of the term (O) of Equation (2) for the first time:

$$TO = \frac{1}{2}.\gamma_1\gamma_2\int_{Z_b}^{Z_m}\frac{\partial}{\partial x}\left\{(D+\varepsilon_x).\frac{\partial C}{\partial x}\right\}.(Y_2^2 - Y_1^2)dz = \frac{1}{2}.\gamma_1\gamma_2\gamma_3\frac{\partial}{\partial x}\left\{(D+\varepsilon_x).\frac{\partial\overline{C_1}}{\partial x}\right\}.(Y_2^2 - Y_1^2).h_1 \quad (8a)$$

where:

γ_3 is the correction coefficient, defined similarly to that in equation (2), calculated as follows:

$$\gamma_3 = \frac{1}{\frac{\partial}{\partial x}\left\{(D+\varepsilon_x).\frac{\partial\overline{C_1}}{\partial x}\right\}.(Y_2^2 - Y_1^2).h_1}.\int_{Z_b}^{Z_m}\frac{\partial}{\partial x}\left\{(D+\varepsilon_x).\frac{\partial C}{\partial x}\right\}.(Y_2^2 - Y_1^2)dz \quad (8b)$$

Calculate the integral of the term (O) of Equation (2) for the second time:

$$T^2O = \frac{1}{2}.\gamma_1\gamma_2\gamma_3\int_{Z_b}^{Z_s}\frac{\partial}{\partial x}\left\{(D+\varepsilon_x).\frac{\partial\overline{C_1}}{\partial x}\right\}.(Y_2^2 - Y_1^2)h_1dz \quad (8c)$$

$$T^2O = \frac{1}{4}.\gamma_1\gamma_2\gamma_3\gamma_4\frac{\partial}{\partial x}\left\{(D+\varepsilon_x).\frac{\partial\overline{C}}{\partial x}\right\}.(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2) \quad (8d)$$

where:

γ_4 is the correction coefficient, defined similarly to that in equation (2), calculated as follows:

$$\gamma_4 = \frac{2}{\frac{\partial}{\partial x}\left\{(D+\varepsilon_x).\frac{\partial\overline{C}}{\partial x}\right\}.(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)}.\int_{Z_b}^{Z_s}\frac{\partial}{\partial x}\left\{(D+\varepsilon_x).\frac{\partial\overline{C_1}}{\partial x}\right\}.(Y_2^2 - Y_1^2)h_1dz \quad (8e)$$

Calculate the integral of the term (P) of Equation (2) for the first time:

$$TP = \frac{1}{2}.\theta_1.\theta_2\int_{Z_b}^{Z_m}\frac{\partial}{\partial z}\left\{(D+\varepsilon_z).\frac{\partial C}{\partial z}\right\}.(Y_2^2 - Y_1^2)dz = \frac{1}{2}.\theta_1.\theta_2.(Y_2^2 - Y_1^2).(D+\varepsilon_z).\frac{\partial C}{\partial z}\Big|_{Z_b}^{Z_m} \quad (9a)$$

Calculate the integral of the term (P) of Equation (2) for the second time:

$$T^2P = \frac{1}{2}.\theta_1.\theta_2.\int_{Z_b}^{Z_s}(Y_2^2 - Y_1^2).(D+\varepsilon_z).\frac{\partial C}{\partial z}\Big|_{Z_b}^{Z_m}dz = \frac{1}{2}.\theta_1.\theta_2.(Y_2^2 - Y_1^2).(D+\varepsilon_z).\left\{\frac{\partial C}{\partial z}\Big|_{Z_b}^{Z_m}\right\}.h \quad (9b)$$

Therefore, the one-dimensional tracer transport equation, obtained directly from the three-dimensional tracer transport equation through the dual approach, is rewritten as follows:

$$T^2I + T^2J + T^2K + T^2L + T^2M + T^2N = T^2O + T^2P \quad (10)$$

$$\begin{aligned}&\frac{1}{4}\alpha_1.\frac{\partial}{\partial t}\{\overline{C}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\} - \frac{1}{2}\alpha_1\overline{C}(Y_2^2 - Y_1^2).\frac{\partial}{\partial t}(Z_s^2 - Z_b^2) - \frac{1}{2}\alpha_{11}\alpha_{12}\overline{C}.(Z_s^2 - Z_b^2).\frac{\partial}{\partial t}(Y_2^2 - Y_1^2) + \\ &\frac{1}{4}\beta_1\beta_2\beta_4.\frac{\partial}{\partial x}\{\overline{UC}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\} - \frac{1}{2}\beta_2\beta_3\beta_6.\overline{UC}.(Y_2^2 - Y_1^2).\frac{\partial}{\partial x}(Z_s^2 - Z_b^2) - \\ &\frac{1}{2}\beta_1\beta_3\delta_1\delta_2\overline{UC}.(Z_s^2 - Z_b^2).\frac{\partial}{\partial x}(Y_2^2 - Y_1^2) = \frac{1}{4}.\gamma_1\gamma_2\gamma_3\gamma_4.(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\frac{\partial}{\partial x}\left\{(D+\varepsilon_x).\frac{\partial\overline{C}}{\partial x}\right\} + \\ &\frac{1}{2}.\theta_1.\theta_2.(Y_2^2 - Y_1^2).(D+\varepsilon_z).\left\{\frac{\partial C}{\partial z}\Big|_{Z_b}^{Z_m}\right\}.h\end{aligned} \quad (11a)$$

Writing Equation (11a) in a more concise form, we obtain:

$$\begin{aligned}&\alpha_1.\frac{\partial}{\partial t}\{\overline{C}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\} - 2\alpha_1\overline{C}(Y_2^2 - Y_1^2).\frac{\partial}{\partial t}(Z_s^2 - Z_b^2) - 2\alpha_{11}\alpha_{12}\overline{C}.(Z_s^2 - Z_b^2).\frac{\partial}{\partial t}(Y_2^2 - Y_1^2) + \\ &\beta_1\beta_2\beta_4.\frac{\partial}{\partial x}\{\overline{UC}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\} - 2\beta_2\beta_3\beta_6.\overline{UC}.(Y_2^2 - Y_1^2).\frac{\partial}{\partial x}(Z_s^2 - Z_b^2) - \\ &2\beta_1\beta_3\delta_1\delta_2\overline{UC}.(Z_s^2 - Z_b^2).\frac{\partial}{\partial x}(Y_2^2 - Y_1^2) = \gamma_1\gamma_2\gamma_3\gamma_4.(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\frac{\partial}{\partial x}\left\{(D+\varepsilon_x).\frac{\partial\overline{C}}{\partial x}\right\} + \\ &2\theta_1.\theta_2.(Y_2^2 - Y_1^2).(D+\varepsilon_z).\left\{\frac{\partial C}{\partial z}\Big|_{Z_b}^{Z_m}\right\}.h\end{aligned} \quad (11b)$$

Equation (11b) preserves critical physical information typically lost in conventional (arithmetical) averaging. The resulting one-dimensional equation is mathematically more general and physically consistent than the equation previously derived by combining dual-approach-based two-dimensional horizontal and vertical systems [11]. Moreover, Equation (11b) demonstrates improved predictive accuracy under complex flow

conditions where classical models often fail to capture key hydrodynamic processes.

When $Y_2 - Y_1 = b \approx Const$ and $Z_s - Z_b = h \approx Const$, over time and space, then $\frac{\partial}{\partial t}(Z_s^2 - Z_b^2) \simeq 0$, $\frac{\partial}{\partial t}(Y_2^2 - Y_1^2) \approx 0$, $\frac{\partial}{\partial x}(Z_s^2 - Z_b^2) \approx 0$, $\frac{\partial}{\partial x}(Y_2^2 - Y_1^2) \approx 0$, if $\frac{\partial c}{\partial z}\Big|_{Z_m} \cong \frac{\partial c}{\partial z}\Big|_{Z_b}$ then the term:

$$2\theta_1.\theta_2.(Y_2^2 - Y_1^2).(D + \varepsilon_z).\left\{\frac{\partial C}{\partial z}\Big|_{Z_b}^{Z_m}\right\}.h = 0$$

and if the coefficients $(\alpha_{ij}, \beta_i, \gamma_i, \delta_i, \vartheta_i) = 1$, this means that when the water level and river width change insignificantly and the change in tracer concentration in the Z direction is also minimal, then from the equation (11b), we obtain the one-dimensional tracer transport equation as follows:

$$\frac{\partial}{\partial t}\{\overline{C}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\} + \frac{\partial}{\partial x}\{\overline{UC}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\} = (Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\frac{\partial}{\partial x}\left\{(D + \varepsilon_x).\frac{\partial \overline{C}}{\partial x}\right\} \quad (12)$$

It is evident that the one-dimensional tracer transport Equation (12) is more general than Equation (21) in [11].

In the case that $(Y_2^2 - Y_1^2)$, $(Z_s^2 - Z_b^2)$ do not change significantly over time and space, that is: $\frac{\partial}{\partial t}\{\overline{C}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\} = (Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\frac{\partial}{\partial t}(\overline{C})$

$$\frac{\partial}{\partial x}\{\overline{UC}(Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\} = (Y_2^2 - Y_1^2).(Z_s^2 - Z_b^2)\frac{\partial}{\partial x}(\overline{UC})$$

Then equation (12) becomes the classic one-dimensional tracer transport equation [2-6], as follows:

$$\frac{\partial}{\partial t}\{\bar{C}\} + \frac{\partial}{\partial x}\{\overline{UC}\} = \frac{\partial}{\partial x}\left\{(D + \varepsilon_x).\frac{\partial \bar{C}}{\partial x}\right\} \quad (13)$$

3. Case Study

Comparing the numerical solution for the salinity concentration of the 1D problem, established by the dual methods above, with the classical method and observed data:

To illustrate the generality of the solute transport Equations (12) established by the dual approach in this paper compared to Equations (21) in [11]. This study examines the phenomenon of salinity intrusion in the Huong River system during the dry season [3].

The Huong River system consists of two branches, Ta-Trach and Huu-Trach, which confluence at Nga-Ba-Tuan, located upstream in the Huong River system. Channels water that subsequently flows through the mainstream of the Huong River towards the Thao-Long barrage at downstream [3] as illustrated in Fig.2.

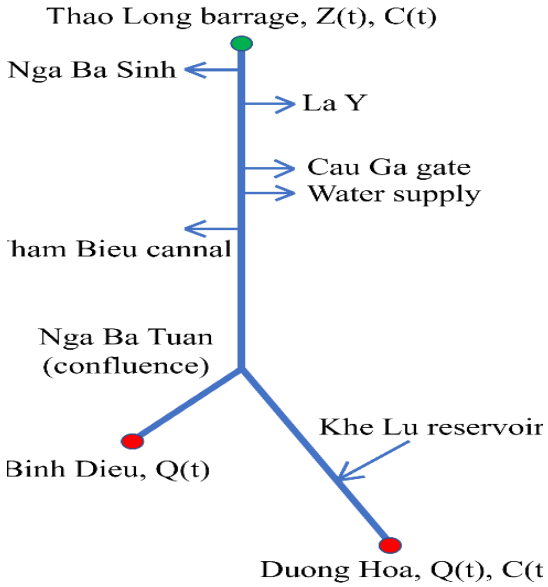

Fig.2. A simplified sketch of the Huong River system [3]

Boundary conditions:

Boundary conditions at upstream: the flow rate is given as $Q_{BinhĐieu} + Q_{DuongHoa} + Q_{KL} = (0.98 + Q_{KL})\ m^3\ s^{-1}$;

Boundary condition at downstream: the tidal water level at the Thao-Long barrage corresponding to a frequency of tidal water level of P = 25%, see Fig.3.

To estimate the optimal size of Khe-Lu reservoir for repelling salinity in the Huong River system, all incremental discharges from the reservoir are considered as outlined in the presented document [3].

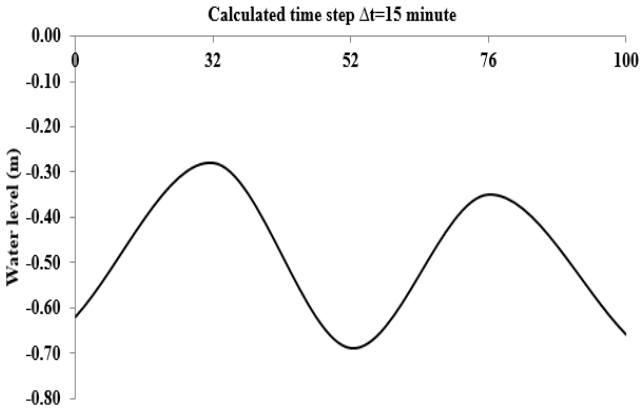

Fig.3. Water level boundary condition at downstream of the Huong River system

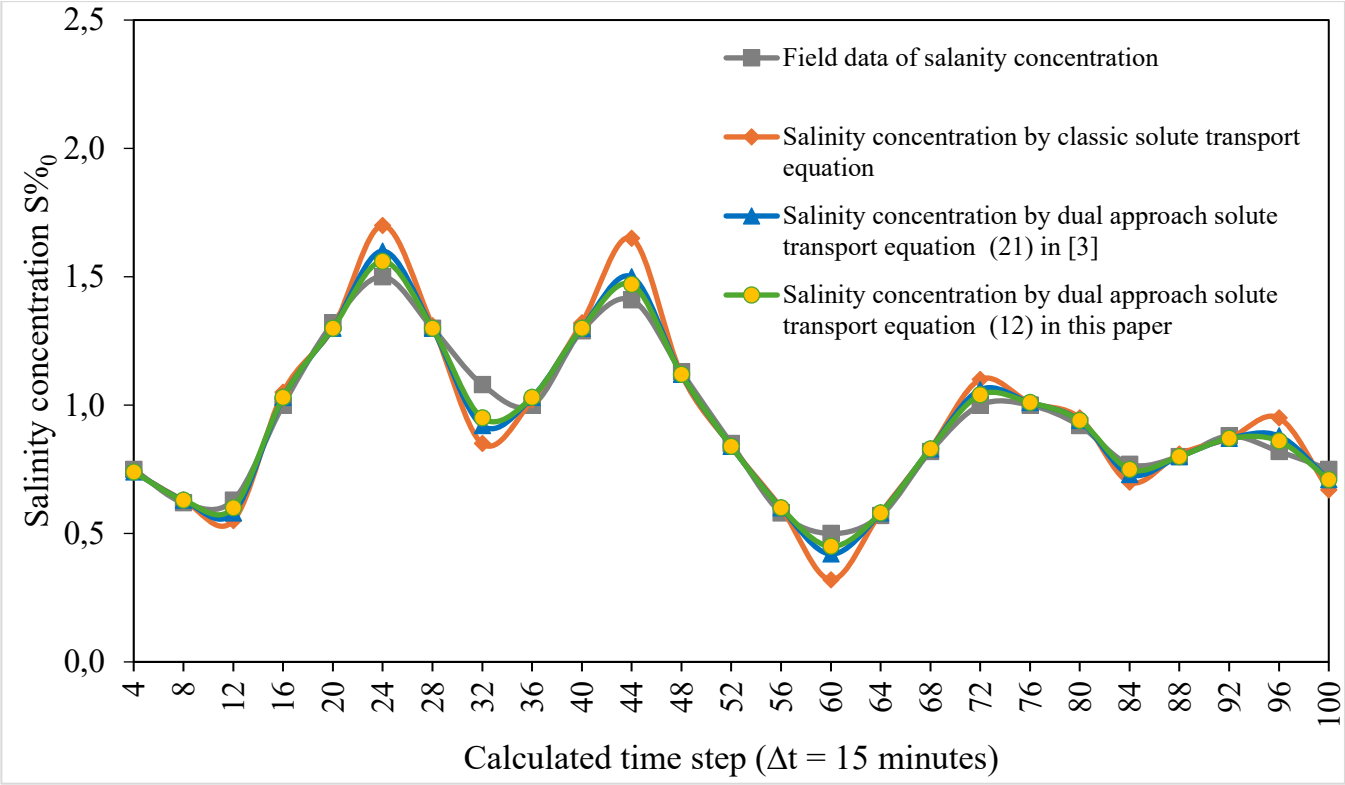

Fig.4. Salinity at La-Y in a tidal period corresponding with the Khe-Lu discharge Q_{KL} = 10.00 m^3

Algorithms and Program:
We apply the finite difference method using the four-point Preissmann scheme, with the weight $\theta = 0.66$, to solve the 1D Saint - Venant equations system [3,5,20]. Then solve the solute transport (equations 12 and 13) in this paper using fractional steps, separating the transport and the diffusion processes. The transport equation is addressed using the characteristic method, while the diffusion equation is solved using a weighted finite difference scheme as described [3,20]. The salinity levels at La-Y, calculated for a Khe-Lu discharge of Q_{KL} = 10.00 m^3 s^{-1}, are presented in Fig.4.

4. Conclusion & Future Work

In this paper, the one-dimensional tracer transport equation in river flow (11b) is established using the dual approach by directly averaging the three-dimensional tracer transport equation across the width and depth of the river, preserving critical physical information typically lost in conventional (arithmetical) averaging. The resulting equation is mathematically more general and physically consistent than the equation previously derived by combining dual-approach-based two-dimensional horizontal and vertical systems.

Equation (11b) for river flow enhances predictive accuracy in environmental assessments, water quality modeling, and tracer experiments, particularly in complex flow conditions where classical models may fail to capture essential hydrodynamics.

Future research should focus on applying the proposed equation in real-case simulations and validating it across a broader range of river geometries and flow regimes.

Declarations

Author Contributions

The Hung Nguyen: Conceptualization, development of the research framework, primary computations, and manuscript preparation;
Hoang Lam Nguyen: Performed computations for specific subsections and contributed to manuscript preparation;
Thi Kim Lien Tran: Assisted with computations in selected subsections and manuscript formatting;
Duc Dung Do**:** Assisted with computations in selected subsections.
All authors have read and agreed to the published version of the manuscript.

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