湖南大学学报（自然科学版） 第 51 卷 第 6 期
Journal of Hunan University (Natural Sciences) 2 0 2 4 年 6 月

Available online at http://jonuns.com/index.php/journal/index Vol. 51 No. 6
June 2024

Open Access Article https://doi.org/10.55463/issn.1674-2974.51.6.31

Stochastic Traffic Flow Algorithm Based on a Macroscopic LWR Model for Real-Time Traffic Prediction

Mehboob Ali Jatoi*[1], Shakeel Ahmed Kamboh[2]

[1]Department of Basic Science & Related Studies, Quaid-e-Awam University of Engineering, Science & Technology, Nawabshah, Sindh, Pakistan

[2]Department of Mathematics & Statistics, Quaid-e-Awam University of Engineering, Science & Technology, Nawabshah, Sindh, Pakistan

* Corresponding author: mehboobali.quest@gmail.com

Received: May 10, 2024 / Revised: May 29, 2024 / Accepted: June 7, 2024 / Published: June 30, 2024

Abstract: Congestion into the traffic flow model is the main factor that affects the free flow of traffic due to the increasing number of vehicles on a daily basis, which is the main drawback of developed countries. There are many disadvantages of the congestion like the increase in accidents, time consumption, air pollution, and many others. The intelligent transportation system depends upon the well prediction of random traffic. As the average flow of traffic is failing due to increasing vehicles. To overcome the disadvantages of a transportation system, an algorithm is proposed to forecast the future state of traffic flow. The proposed method can be used to predict the traffic flow of any section of congested road to predict the traffic. A section of road is considered for real-time traffic data for a week from 7:00 to 19:00 with one hour of time interval. Shiparo-Wilk test with 95% confidence level is applied for the normality of the data which fails to be a normal. Then various statistical tests for the distribution are applied using the MATLAB distribution fit tool box, and different distributions like normal, lognormal, Weibull, exponential, and gamma distributions are applied for the best fit over the collected data, and a lognormal distribution with less standard error is the best fitted distribution over the collected data. For stochastic traffic flows, a forcing function with incorporation of initial density and flow using Brownian motion is studied for the proposed model, and a lognormal distribution is replaced by Brownian motion, which shows randomness in to the model. Godunov's numerical method was used for the proposed model with discontinuity and was incorporated using the Riemann problem solver, and the algorithm with randomness followed the collected data, which showed that the model had good predicted performance.

Keywords: Macroscopic Lighthill, Whitham and Richards (LWR) model, randomness, Shiparo-Wilk test, distributions, Godunov's method

基于宏观 LWR 模型的随机交通流实时预测算法

摘要：在交通流模型中，拥堵是影响交通畅通的主要因素，因为每天行驶的车辆数量都在增加，这是发达国家的主要缺点。拥堵有很多缺点，例如事故增加、时间浪费、空气污染等等。智能交通系统依赖于对随机交通的良好预测。由于车辆增加，平均交通流量正在下降。为了克服交通系统的缺点，提出了一种预测未来交通流状态的算法。所提出的方法可用于预测任何拥堵路段的交通流量以预测交通状况。考虑一段路的实时交通数据，为期一周，从 7:00 到 19:00，时间间隔为 1 小时。对不符合正态的数据应用置信度为 95% 的 Shiparo-Wilk 检验。然后利用 MATLAB 布拟合工具箱对分布进行各种统计检验， 并应用正态分布、对数正态分布、威布尔分布、指数分布和伽马分布等不同分布对收集的数据进行最佳拟合，标准误差较小的对数正态分布是收集数据的最佳拟合分布。对于随机交通流，研究了采用布朗运动的初始密度和流量的强制函数，并用布朗运动代替对数正态分布，这在模型中表现出

随机性。对于具有不连续性的所提出的模型，采用了 Godunov 的数值方法，并使用了黎曼问题求解器，随机性算法遵循收集的数据，表明该模型具有良好的预测性能。

关键词：宏观 Lighthill、Whitham 和 Richards (LWR) 模型、随机性、Shiparo-Wilk 检验、分布、Godunov 方法

1. Introduction

Traffic congestion in urban areas faces many challenges, and a traffic flow model that accurately predicts traffic conditions may be useful in responding to these challenges. The deterministic model is used to capture traffic flow conditions; however, there are many stochastic factors like irregular driver behavior, weather conditions, road conditions, and special events. In a congested situation of traffic, a huge number of negative effects may occurs like accidents, delays in traffic, waste of time, increased travel time, reduced efficiency of the traffic system, and environmental effects like increase in fuel consumption, greenhouse gas emissions, air pollution, noise, and many others on the traffic and surrounding environment. Congestion over the road in traffic flow is due to the maximum increase in the number of vehicles on the road that pass a specific expanse per unit time.

During the external effects on traffic, it is difficult to predict the deterministic model for the traffic flow. To predict traffic flows, an algorithm was developed for a stochastic traffic flow model.

Deterministic model:

- Initial circumstances were used to investigate the model output and parameter values.
- For well-defined linear models, a single output is generated for a single input, whereas for nonlinear models, several outputs are available.
- Different numerical methods like finite difference method (FDM) and finite Element method (FEM) are used to solve the differential equation.
- Model can be described at different levels of temporal variation like steady state, unsteady state, and dynamic

Stochastic model:

- It has inherent randomness; thus, distinct results with the same initial values that represent the randomness
- Because behavioral characteristics fluctuate, the stochastic model accounts for uncertainties.
- A random number is generated in the simulation of a stochastic model to execute trials. This type of simulation is called a Monte Carlo simulation.
- Stochastic model to estimate the probability distribution.
- A random component of a distribution is an input used by a stochastic model, and its result is another distribution.

The main source of variation in the model result was the variation in the input variables. When the variation is negligible or non-existent, there is no need to apply stochastic, deterministic approaches to describe the traffic flow. The common factors that play an important role in a stochastic traffic flow model are headway, spacing, and velocity; thus, it is significant to study the stochastic characteristics in traffic evolution. [1] According to [2], the overall traffic flow can be divided into three categories. Microscopic, mesoscopic, and macroscopic traffic flow model. Microscopic models are the study of driver's action and the communication between individual automobiles at a very great level of point. The method is used for each vehicle using driver behavior, car following, and the choice of path for a vehicle by deriver decision. [3]. Macroscopic models are the most used and powerful type of traffic flow model, and they describe traffic evolution on the basis of average traffic flow. The model focuses on the main variables, such as the flow of vehicles (vehicles/hr.), density (vehicles/km), and speed (km/hr.) of vehicles. Mesoscopic models combine the microscopic and macroscopic models and are also called hybrid models. The three main mesoscopic models are the headway distribution, cluster, and gas kinetic continuum models. The Lighthill, Whitham and Richards (LWR) model is a traditional macroscopic model proposed by Lighthill and Whitham and Richards. An LWR-based macroscopic model was described in [4]. An alternative explanation of the evolution of traffic flow is provided by focusing on the transmitting and receiving functions in each cell and lane. The macroscopic model is widely used to obtain average traffic flows [5]. The LWR model is used to describe the basic structure/model of many macroscopic traffic flow models, including the PW model (Payne [6]; Whitlam, [7], and Zhang's [8]) model. Normally, in all macroscopic traffic flow network models, there are three main parameters used to define the traffic network, and the same are parameters can be used for stochastic traffic flow modeling by considering the randomness in parameters:

- Flux denotes the flow of traffic q (vehicles/hr.),
- Density ρ (vehicles/km)
- Average speed vs (km/hr.).

The fundamental flow relationships between these parameters are defined by the following equation:

$$q = \rho v \qquad (1)$$

The conservative law in a macroscopic traffic flow model without any ramping gradient can be defined as, which is also called the equilibrium LWR model:

$$\frac{\partial}{\partial t}\rho(x,t)+\frac{\partial}{\partial x}Q(x,t)=0 \qquad (2)$$

Hence, the LWR model is a first-order partial-differential equation for equilibrium flow. In addition, there is another traffic flow model as second order based on fluid dynamics proposed for enhancing the LWR model [6]. Therefore, for the LWR model, the stochastic model can be discussed by adding a stochastic forcing function in equation that incorporates all the effects that are applied to (x,t). Nowadays, traffic flow requires a probabilistic model to predict the traffic flow situation. This probabilistic traffic flow can be optimized using a "free-flow" model in which traffic with no discontinuity by any characteristics by traffic flow parameters like road traffic lights, stop signs, bifurcation or traffic jamming, and driver speed [7]:

$$V=V_{\min}\left(1-\frac{\rho}{\rho_{\max}}\right) \qquad (3)$$

The actual flow with low density of traffic was reported in [8], where real traffic flow data collected from several highways and the time breaks between vehicles on the highway were characterized by a Gaussian-exponential mixture model. As a result, the probability density function PDF for the time interval time, similarly, the vehicular arrival time with maximum density profile is commonly modeled as a Poisson process, hence to estimate the average number of vehicles inside the segment SD Little's formula [9] is used. The whole scenario of the traffic flow was divided into three different ways [10], the data on traffic flow should be nonnegative and can be found proper probability distribution, secondly the complex stochastic property of the traffic flow leads the non-stationary variance, and at last, the multistep-ahead prediction of the traffic flow often has poor performance. For this purpose, a gamma distribution-based time-series model is used. The model to predict the probability distribution function of the traffic flow in real time and same Gamma distribution present the stochastic properties of non-negative-valued traffic flow. The property of path was eliminated in [11] following model restriction using the LOGIT model. the stochastic interpretation of experimental information for vehicles on road was represented in [12] using equilibrium and non-equilibrium condition and results to revise the classical kinetic models. According to [13], the analysis of traffic characteristics implements vehicular flow inside a road segment using a proper traffic flow model and describes the feasibility of running a certain class of application vehicular flow for both light and high vehicular flow using the different factors of traffic flow like the number of vehicle and time present vehicles in a region and present the two type of model, which are the free-flow model and queuing-up model. The numerical solution algorithm for reliability-based stochastic traffic model was described in [14] using day-to-day demand in traffic model based on stochastic travel time using demand variation, also considered the traveler's perception error on travel time and solved the model. For stochastic travel demand, travel time is considered as a random variable. A microscopic stochastic traffic flow model was proposed in [15], a Markov chain with an interaction law was developed, and a stochastic process was proposed to consider spic-flip and spin-exchange traffic flows for entering and leaving the vehicle and interacting. The model was simulated by kinetic Monte Carlo simulation and examined for extreme and typical traffic flow scenario and predict traffic for freeway traffic geometry. According to [16], flow-density relationship may be affected by stochastic variation in driving behavior and both the hysteresis transition and wide scattering can be developed by multiclass first-order model using stochastic setting in parameters of the model, using simulation over the real data collected on freeway and consistent the proposed model using some numerical experiments and described the proposed model is more reliable as compared to the fundament LWR model. The stochastic traffic model was simulated in [17] to estimate the capacity of road using macroscopic traffic flow model, the model has the capacity to estimate the covariance in the distribution of traffic flow, the capacities are classified in summer and winter seasons by collecting traffic data using video camera, also examine the traffic flow travel time. The stochastic traffic flow model was described in [18] and a stochastic partial differential equation was considered to describe the behavior of real traffic on highways using the simulation for the real traffic flow and the model has the better capability to predict the traffic behavior and can be better improved using ramps and changing the number of lanes. The total number of vehicles that pass from position x in time t can be represented by Poisson integration. A stochastic traffic flow model was represented in [19] for VANETs with a signalized urban road. The model was composed using a fluid model and stochastic model and computed the traffic density randomly to check the random behavior of vehicles. A stochastic traffic flow for highways was proposed in [20] by combining cellular automata and microscopic traffic flow models, with computational efficiency for both models by considering continuing in space. The proposed model can be used to develop traffic flow scenarios with heterogeneity effects in traffic steam. A macroscopic stochastic traffic flow model was presented in [21] with an offline regulation algorithm and developed the Godunov numerical algorithm for the stochastic prediction of traffic flow based on the model and showed that the model has better prediction accuracy of traffic flow. A new stochastic model was developed in

[22] by extending the Godunov scheme based on a queuing theoretical approach by adding a noise deterministic traffic flow model by considering probabilistic car vehicle inter-crossing times, and the result showed the non-negativity of traffic variables (densities). A traffic flow model was proposed in [23] to capture the stochastic nature of route traffic flow using the conditional probability distribution and considered the equilibrium traffic flow and estimate characteristic such as mean and variances of route traffic flow from the simulation of the model; the model was highly flexible for computational process. The main aspect of the stochastic traffic model was described in [24] for vehicles over the road section, and it was defined on continuous time on a distinct space using a Markov process for the Mesoscopic traffic flow model. The novel, flexible, and robust traffic flow cellular model was developed in [25], considering two important keys for the drivers' behavior and expecting that vehicles were in front of them. In addition, to understand how a specific vehicle gets around, the stochastic technique was considered using the probability density function of the beta distribution to mode the deriver behavior. This model was used to calculate the desire velocity of a vehicle. The stochastic parameters that are driven from realistic driver behavior were used in [26], which implemented on a discrete fine-grained agent-based model and combined into a deterministic Aw-Rascle system of equations and applied first-order and second-order traffic model, using second order Lax-Wendroff scheme. The model was presented in [27] to incorporate the interaction and stochastic behavior unique to each vehicle in traffic flow simulation. In a platoon-based Lagrangian coordinate system, a first-order macroscopic model with stochastic advection, showed how vehicle characteristics influenced surrounding vehicles and how local vehicles adjusted at a critical density. the congested situation of traffic flow in urban areas was reduced in [28] by considering the stochasticity of microscopic traffic models and deterministic models by limiting the road length and dividing the section into crossing sections. The model was optimized using a receding–horizon scheme, and the model was capable of understanding and oversaturated traffic conditions. According to [29], the speed and density of a couple of vehicles for real traffic data were predicted using an unscented Kalman filter. The model was 20% more improved in terms of predicting the traffic speed and density than the deterministic model. The proposed method also attempted to predict the jam density of traffic in the situation of stop-and –go traffic. The data-driven stochastic car-following model was developed in [30] using a database car-following model, and the acceleration of the vehicle was considered directly from the data. In most cases, the acceleration value is distributed according to the Laplace distribution, and missing data can be obtained directly using interpolation. In addition, using this model, the data obtained are the same as in reality. The model is used to predict the next position of a vehicle at a time, and vehicle acceleration is also founded. An alternative stochastic traffic flow model was developed in [31] for the fundamental diagram of traffic flow using a minimum number of vehicle parameters using a mesoscopic traffic flow model. This model was not only used for the fundamental diagram of traffic flow parameters but also for the uncertainty and variance of fundamental diagram. The stochastic model was presented in [32] using Lagrangian coordinates to predict the traffic flow. The drivers can vary free flow speed and minimum distance between cars. Another model was presented in [33] to predict the behavior of vehicles effected while automated vehicles using sensor, and it also described the effect of lane changing prediction by combining deterministic and probabilistic forms. The effects of lane changing is simulated using MATLAB. The model is guaranteed for safe lane-changing behavior. The stochastic features in urban traffic flow were described in [34] using the stochastic connection flow model for signalized traffic flow network with reservations with four link state mode and probability of each link was found based on stochastic link state by considering the data from microscopic traffic flow model simulator; the model had good estimation on the ink state uncertainties with the signalized traffic flow model. A methodology to simulate the fundamental diagram as a stochastic traffic flow model which applied to real traffic flow model to capture the relation between the flow density of the stochastic traffic flow model was described in [35] using the collected data and statistical analysis shows overall quality of distribution fitted stochastic process is acceptable. A stochastic traffic flow model was presented in [36] in terms of Lagrangian coordinates to consider the variability in driving behavior, which is recorded by driver-specific speed-spacing relationships or parameter uncertainty. It resulted in trajectories of both real-traffic flow dynamics and oscillations in the path of stochastic traffic flow models. The proposed method is suitable for real-time applications that use recursive data. A novel model was used in [37]-[38] to describe the stochasticity over the macroscopic traffic flow using the stochastic speed-density relation. The impact of stochasticity on traffic flow was investigated for upstream and downstream bottlenecks. The model showed that experimental spatiotemporal patterns could be produced using macroscopic model using stochasticity speed-density relation. A stochastic traffic flow model for mixed traffic flows was described in [17] using Lagrangian coordinates; it accounted for the varied behaviors of human drivers in conjunction with automated and human-driven vehicles. Also, the first- and second-order approximations of a stochastic model were created that described the mean and covariance dynamics, respectively, for various configurations of

randomly driven human and automated vehicles in a traffic stream. The proposed model can be used to investigate the interaction between human and automated driving. The stochastic Lighthill-Whitham Richards model was developed in [39] by considering the heterogeneity and solved using Lagrangian coordinates. The model was used to investigate the effect of driver heterogeneity on the macroscopic relation of traffic flow parameters both analytically and in simulation and to investigate that the static and dynamic macroscopic traffic flow characteristic like speed and flow rate at the bottleneck, are consistent with a deterministic model and the jam density, which is the harmonic mean of distribution. Also, the relation between automation vehicles and dispersion vehicles flow rate was discussed to maximize vehicle at the bottleneck by reducing the capacity drop effect. Based on the literature review, most researchers use a microscopic traffic model, while we proposed to use a macroscopic model to show the actual randomness in the model. For the randomness in congested and random situations, the prediction of traffic flow is not an easy task due to stochastic factors and discontinuity in traffic, like continuously increasing road vehicles, congestion of roads, weather conditions, and road changes. Conservation of traffic flow under macroscopic traffic conditions does not incorporate stochastic factors. The main aim of this study is to develop an algorithm that incorporates stochastic factors with randomness in traffic flow using the online calibration of traffic data to predict the future state of traffic flow over a road section.

2. Model Formulation

Let us consider a homogeneous road section with length L, where the flow at the boundaries of the road section is the deterministic LWR model:

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial Q(x,t)}{\partial x} = 0 \qquad (4)$$

With $Q(x,t) = \rho(x,t)v(x,t)$.

The stochastic model captures the variations/changes in the traffic flow parameters. In a stochastic model, the density and velocity can be defined as the partial derivative of the expected number of vehicles. $\rho_e(x,t) = \frac{\partial}{\partial x}\rho(x,t)$ and $v_e(x,t) = \frac{\partial}{\partial x}v(x,t)$, where

$$\rho_e(x,t) = \rho(x,0) + \Delta x \bar{Q}(x,t) + W(x,t) \qquad (5)$$

With $\bar{Q}(x,t)$ is the deterministic flow $\rho(x,0)$ is the initial density, and *W(x, t)* is the Brownian motion that captures the stochastic behavior of the model; hence, the LWR model becomes with source term,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}Q(\rho) = u(\rho_e, x, t) \qquad (6)$$

where $u(\rho_e, x, t) = \rho(x,0) + \Delta x \bar{Q}(x,t) + W(x,t)$.

For the solution of the stochastic partial differential equation (SPDE), the value of $u(\rho_e, x, t)$ must be known over (x,t). Where $\rho(x,0)$ *and* $Q(x,t)$ $\rho(x,0)$ *and* $Q(x,t)$ are the initial density at t = 0 and flow, respectively. $W(x,t)$ is the Brownian motion, which captures randomness in traffic flow with distribution over the collected data. The proposed model equation with the stochastic partial difference equation (SPDE) is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}Q(\rho) = \rho(x,0) + \Delta x \bar{Q}(x,t) + W(x,t) \qquad (7)$$

3. Methodology:

3.1. Area Selection and Data Collection

For the collection of data with congestion, which is affected by the stochastic factor, a section of road is selected figure 1, which consists of the inflow and outflow of traffic vehicles. The total number of paths with inflow and outflow is 18. The inflow with ramp in and out is neglected as the inflow or outflow at the path location. The data is collected manually for all paths for a week August 23-29 from 7:00 to 19:00 and data is divided in to one-hour intervals. The complete methodology is defined as follows: Deterministic macroscopic LWR model is considered to model the Stochastic model, A homogeneous section of road with congestion is considered for the collection random data for the model, Data Collection and analysis of collected vehicle flow, Normality test for collected data, and fitting a suitable probability distribution over the Collected data for stochasticity. To consider the Godunov numerical scheme, we developed an efficient algorithm using MATLAB to predict the traffic flow for the model solution.

Fig. 1 Sattelite view of the section of road, August 23, 2023 (developed by the authors)

4. Model solution

After collecting the traffic flow data over all paths, the data is statistically analysis as mean, standard deviation, skewness, range, week wise and day wise to analyze all paths with an amount of traffic at the considered time intervals. Then, all data over all paths were tested to determine the normality of the collected data. The Shiparo-Wilk test is used to varify the normality of the collected data; thus, the test fails for the normality of the traffic data for all paths (Table 1). Then, to incorporate the stochastic behavior of the traffic flow, the normal probability distribution, exponential probability distribution, Weibull probability distribution, lognormal probability distribution, and gamma probability distribution were applied using the MATLAB distribution tool box with 95% confidence interval, and it was found that all data on all paths of the stochastic behavior of the traffic data is Lognormal. Also, the standard error is less as the standard error of the other distribution, Fig. 2 shows the lognormal distribution with standard error for path S1.

Table 1 P-values for Shiparo-Wilk test (developed by the authors)

Paths	p-value
S1	0.002123
S2	0.0001275
S3	0.005329
S4	0.01141
S5	0.03559
S6	0.003883
S7_in	0.01542
S7_out	0.0001254
S8	0.00838
S9	0.0126
S10_in	0.005835
S10_out	0.000364
S11_in	1.317e-7
S11_out	0.002022
S12_in	4.463e-7
S12_out	0.006629
S13	0.007408
S14	0.04417

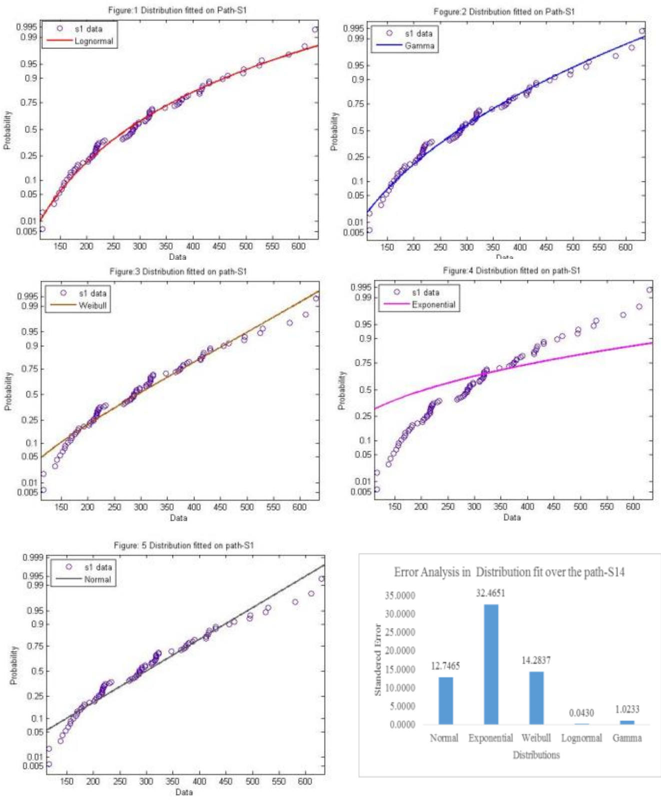


Fig. 2 Distribution fit for the data on path S1 (developed by authors)

Similarly, the other basic parameters like critical density, velocity profile, and minimum density over all the paths, which shall be incorporated into the proposed model to solve the LWR model, are set using the collected data over all the paths and are shown in Fig. 3; and Fig. 4 shows the Brownian motion profile.

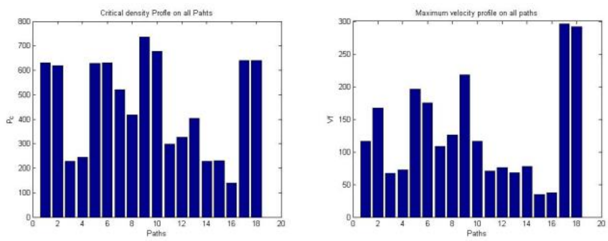


Fig. 3 Density and velocity profiles (developed by the authors)

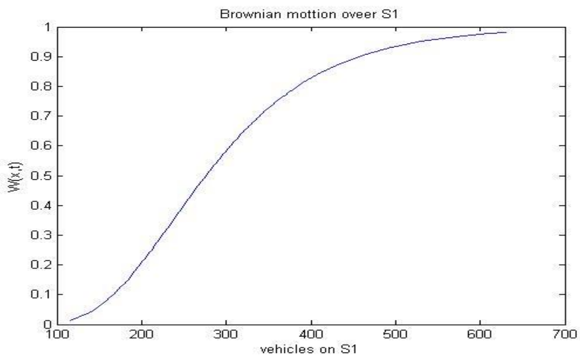


Fig. 4 Brownian motion profiles (developed by the authors)

Due to the nonlinearity of the traffic flow behavior, the traffic flow can be modeled as a stochastic partial differential equation. Due to the nonlinearity into traffic, the flux function is nonlinear and has no partial derivative exts but can be modeled as an integral equation and shows a unique solution. These types of solutions generate shock waves under discontinuous conditions. In result, if the vehicle flow increases, then the speed decreases and congestioni is created, which also generates shocks in physical form, and the flow of vehicles is in discontinuous form. Hence, due to this discontinuity, the partial differential equation can be rewritten as a stochastic partial differential equation. The direct solution of these types of stochastic partial differential equations does not exist because they are discontinuous. Godunov's scheme is a numerical scheme to approximate the solution under these discontinuous conditions. The model stochastic partial-differential equation with the forcing function can be defined as

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}Q(\rho)=u(\rho_e,x,t) \quad (8)$$

is the partial differential equation as the conservation equation with the forcing term, where $u(\rho_e,x,t)$ is the forcing term, $Q(\rho)$ as the flux function, and is the nonlinear function. Thus, if the forcing function is zero $u(\rho_e,x,t)=0$, then the stochastic partial differential equation can be converted into a homogeneous partial differential equation; for the non-zero forcing function, the stochastic partial differential equation is a nonhomogeneous partial differential equation.

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}Q(\rho)=0 \quad (9)$$

Simple finite-difference methods are not capable of finding the solution of the partial-differential equation due to the nonlinear flux function. Hence the Godunov's scheme is the most powerful method to find the approximate solution of the partial differential equation due to discontinuity for both homogeneous and nonhomogeneous partial differential equations. In the stochastic partial differential equation, the density is considered a piecewise constant, and its exact solution can be calculated using the Riemann problem at the boundaries.

Let us consider the LWR model and the solution of the LWR model equation can be found using the two methods, one using fundamental relation and other one traffic characteristics at an initial time referred as initial data on density and is written as $\rho(x,0)=\rho_0(x)$ is the function of x.

Let the $\rho(x,0)=\rho_0(x)$ is continuous function of *x*, then the free-density flow relation is given by $Q(x,t)=\rho v,\ v>0$, then the LWR conservation equation implies

$$\frac{\partial \rho}{\partial t}+\frac{\partial Q(\rho)}{\partial}=0$$

$$\Rightarrow \frac{\partial \rho(x,\mathrm{t})}{\partial t}+v\frac{\partial \rho(x,t)}{\partial}=0 \quad (10)$$

is the linear advection equation, which is well-posed and obtains the

$$\rho(x,t)=\rho_0(x-vt) \quad (11)$$

Hence, the initial density remains the same and the space shifts at speed *v* over time.

Let the densities (x_1,t_1) lie at the origin, then the corresponding characteristic equation is $x_0=x_1-vt_1$, then the slope of the characteristic lines are $1/v$ due to the orientation of the axes. Similarly, let the $Q(\rho(x,t))$ is the flow density relation, then conservation equation can be defined as

$$\frac{\partial \rho(x,t)}{\partial t}+Q'(\rho(x,t))\frac{\partial \rho(x,t)}{\partial x}=0 \quad (12)$$

Where $Q'=\frac{dQ}{dx}$, then the slope of characteristic lines represents $1/Q'(\rho(x,t))$.

The position at which the characteristic lines of the conservation equation intersect is to consider the more densities at one point that creates the decrease into the velocities is called the shock waves, and the point of intersection is called the discontinuous point. Let $\bar{x}(t)$ is the discontinuous point at time t and $[x_{1,}x_2]$ is the section of road containing $\bar{x}(t)$, the conservation equation is

$$\frac{d}{dt}\int_{x_1}^{\bar{x}(t)-}\rho(x,t)dx+\frac{d}{dt}\int_{\bar{x}(t)+}^{x_2}\rho(x,t)dx \quad (13)$$
$$=Q(\rho(x_1,t)-Q(\rho(x_2,t)$$

Where $\bar{x}(t)-$ and $\bar{x}(t)+$ are the left and right side of the $\bar{x}(t)$

By applying the Leibniz rule to the left side, the above equation can be rewritten as

$$\rho(\bar{x}(t)-,t)\frac{d}{dt}\bar{x}(t)+\int_{x_1}^{\bar{x}(t)-}\frac{\partial}{\partial t}\rho(x,t)dx-$$
$$\rho(\bar{x}(t)+,t)\frac{d}{dt}\bar{x}(t)+\int_{\bar{x}(t)+}^{x_2}\frac{\partial}{\partial t}\rho(x,t)dx= \quad (14)$$
$$Q(\rho(x_1,t)-Q(\rho(x_2,t)$$

Since $\rho(x,t)$ is differential able away from the discontinuous point, the integral is zero; thus,

$$\rho(\bar{x}(t)-,t)\frac{d}{dt}\bar{x}(t)-\rho(\bar{x}(t)+,t)\frac{d}{dt}\bar{x}(t) \quad (15)$$
$$=Q(\rho(\bar{x}(t)-,t)-Q(\rho(\bar{x}(t)+,t)$$

$$\Rightarrow \frac{d}{dt}\bar{x}(t)=\frac{Q(\rho(\bar{x}(t)-,t)-Q(\bar{x}(t)+,t)}{\rho(\bar{x}(t)-,t)-\rho(\bar{x}(t)+,t)} \quad (16)$$

is the shock wave that forms the queue for vehicles (congestion), and is considered the flow before (left side) and after (right side) the discontinuity point, and are shown as ρ_L and ρ_R for left and right side flows from the discontinuity point, respectively. There are two cases for the flow at discontinuity point (discontinuous initial data) as $i)\ \rho_L < \rho_R \quad ii)\ \rho_L > \rho_R$ shown in Fig. 5 and is referred to as the Riemann problem.

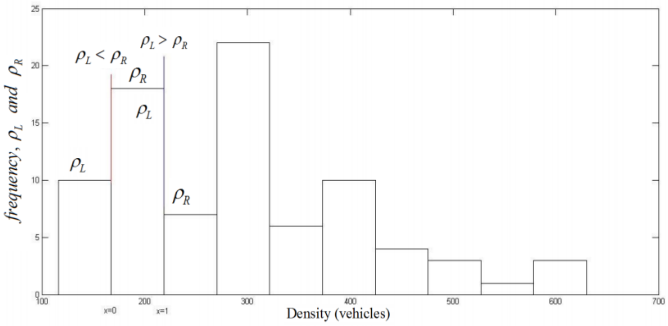


Fig. 5 Differentiating the left and right density (developed by the authors)

In the case of discontinuous initial flow, let $Q(\rho(\bar{x}(t)-,t) = Q(\rho_L),\ Q(\rho(\bar{x}(t)+,t) = Q(\rho_R)$ then

$$\frac{d}{dt}\bar{x}(t) = \frac{Q(\rho_L)-Q(\rho_R)}{\rho_L-\rho_R} = S \qquad (15)$$

Where S is the shock-wave speed constant, which is called the Rankine–Hugoniot (R-H) jump condition. As S is a constant, the same can be rewritten and integrated as follows:

$$\frac{d}{dt}\bar{x}(t) = S \ \Rightarrow\ \bar{x}(t) = S\,t \ \Rightarrow S = \frac{\bar{x}(t)}{t}$$

For the solution of Riemann problem with the initial data can be found using Lax-Entropy condition [27]:

i) $\rho_L < \rho_R$

$$\rho(x,t) = \begin{cases} \rho_L\,, \text{ if } & \dfrac{\bar{x}(t)}{t} < \dfrac{Q(\rho_L)-Q(\rho_R)}{\rho_L-\rho_R} \\ \rho_R\,, \text{ if } & \dfrac{\bar{x}(t)}{t} > \dfrac{Q(\rho_L)-Q(\rho_R)}{\rho_L-\rho_R} \end{cases}$$

ii) $\rho_L > \rho_R$

$$\rho(x,t) = \begin{cases} \rho_L\,, & \text{if } \dfrac{\bar{x}(t)}{t} < Q'(\rho_L) \\ Q'^{-1}, & \text{if } Q'(\rho_L) < \dfrac{\bar{x}(t)}{t} < Q'(\rho_L) \\ \rho_R\,, & \text{if } \dfrac{\bar{x}(t)}{t} > Q'(\rho_L) \end{cases}$$

Therefore, the value of $\rho(x,t)$ shall be corporate into the Godunov's scheme. The flow using Godunov's scheme is then defined as

$$Q(\rho_L,\rho_R) = \max_{\rho_R<\rho<\rho_L} Q(\rho),\ if\ \ \rho_L > \rho_R$$

$$Q(\rho_L,\rho_R) = \min_{\rho_L<\rho<\rho_R} Q(\rho),\ if\ \ \rho_L < \rho_R$$

Therefore, by considering the Riemann solver, the left and right densities, and by function incorporating the SPDE, the approximation solution can be obtained using the Godunov's numerical scheme. Godunov's method is a conservative numerical technique for solving partial differential equations. The conservative variables are to be considered piecewise constants over each cell, and each step and time is evaluated based on the precise Riemann Problem solution at the intercellular borders. The Godunov's method is used for both homogeneous and non-homogeneous conservation laws by discretizing the space region [0,L], where L is the length of the road divided into N cells as $x_0, x_1, x_2, ..., x_{N-1}, x_N$ with length of cells $\Delta x_i = x_i - x_{i-1}$ and the time domain into an M time interval.

The model equation with forcing factor as $u(\rho_e,x,t) = \rho(x,0) + \Delta x\,Q(x,t) + W(x,t)$ is

$$\frac{\partial\rho}{\partial t} + \frac{\partial Q}{\partial x} = \rho(x,0) + \Delta x\,Q(x,t) + W(x,t) \qquad (16)$$

where $W(x,t)$ is the Brownian motion stochastic term. Using the central difference technique, the model discretization is

$$\frac{\rho(x,t+\Delta t)-\rho(x,t)}{\Delta t} + \frac{Q(x+\Delta x,t)-Q(x,t)}{2\Delta x} = \rho(x,0)+\Delta x Q(x,t)+W(x,t)$$

After simplification,

$$\rho(x,t+\Delta t) = \rho(x,t) + \Delta x\left(\begin{array}{l}\rho(x,0)+\Delta xQ(x,t)+W(x,t)- \\ \dfrac{Q(x+\Delta x,t)-Q(x,t)}{2\Delta x}\end{array}\right) \qquad (17)$$

Using the Godunov numerical scheme equation (3), we obtain

$$\frac{\partial\rho}{\partial t} + \frac{\partial Q}{\partial x} = u(\rho_e,x,t)$$

$$\rho_i^{j+1} = \rho_i^j + \frac{\Delta t}{\Delta x}\left(\begin{array}{l}Q(\rho_{i-0.5}^j)-Q(\rho_{i+0.5}^j)+ \\ u(\rho_i^i,x_i,t_j)\Delta x_i\end{array}\right) \qquad (18)$$

Where flow Q is $Q(\rho(x,t)) = v\rho(v(x,t))$ can be calculated using the Greenburg model

$$v(\rho) = v_f\left(1-\frac{\rho(x,t)}{\rho_{jam}}\right)$$

Hence the following algorithms for homogeneous and non-homogeneous stochastic partial differential equations using the Godunov's method can be adopted

$$\rho_i^{j+1} = \rho_i^j + \frac{\Delta t}{\Delta x_i}\left(\rho_i^0 + 4(Q_{i-1/2}^j - Q_{i+1/2}^j) + W_i^j) - \frac{Q_{i+1}^j - Q_i^j}{2\Delta x_i}\right)$$

to approximate the solution. An algorithm for the Godunov's scheme using the Riemann problem solver. To incorporate the forcing function into conservation law, the system can be represented as a nonhomogeneous conservation law. The initial density can be represent as $\rho(x,0)$ at $t=0$, the forcing function is given by

$$u(\rho_e, x, t) = \rho(x,0) + \Delta x Q(x,t) + W(x,t)$$

with Δx as length of road, which is approximately 4 km and $Q(x,t) = \rho v$, where $v(\rho(x,t)) = v_f\,(1 - \frac{\rho}{\rho_{jam}}),\ \rho_{jam} = \max(\rho(x,t))$, and $W(x,t)$ is the Brownian motion, which is considered by the model fit distribution, which is a lognormal distribution; hence, Brownian motion can be incorporated as a lognormal distribution because the data are fitted by lognormal. Hence, Algorithm 2 below represents Godunov's method for non-homogeneous conservation law by incorporating the forcing function.

4.1. Algorithm. Godunov's Method for Non-Homogeneous Partial Differential Equation

Step1. Start. Let density ρ_i^j show the mean density across cell i at that moment t_j. When j = 0, the initial density for ρ_i^0 is

$$\rho_i^0 = \int_{x_{i-1}}^{x_i} \rho(x,0)dx \quad \forall i = 1, 2 = 3, \dots, N$$

Step 2. Godunov's Decomposition: by neglecting the source term, let's consider $\rho_i^{j,-}$ and $\rho_i^{j,+}$ mean value of density ρ in the left and right boundaries from a discontinuous or jumping point.

Hence

$$\rho_i^j = \frac{\rho_i^{j,-} + \rho_i^{j,+}}{2}$$

$$\frac{Q(\rho_i^{j,+}) - Q(\rho_i^{j,-})}{\Delta x_i} = u(\rho_i^j, x_i, t_j)$$

Step 3. Riemann Solver. Let $\rho_{i-1/2}^j$ and $\rho_{i+1/2}^j$ denotes the average value of the density ρ over the time interval $[t_j, t_{j+1}]$ at the left and right boundaries of cell *i*, respectively. The value of $\rho_{i-1/2}^j$ and $\rho_{i+1/2}^j$ is obtained using the Riemann problem equation.

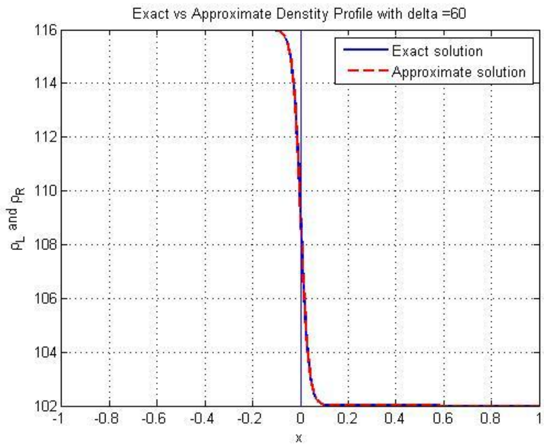
Step 4. In accordance with step 2, the specified boundary density is updated. The speed-density relation function is used to find the central traffic flow. The density is updated in subsequent time intervals in accordance with conservation law. Additionally, averaged over each cell in the subsequent time interval, the state variable becomes a piecewise constant. Approximation. After the update, $j = j+1$ and again move to step2. The updated density is then added to step 2.

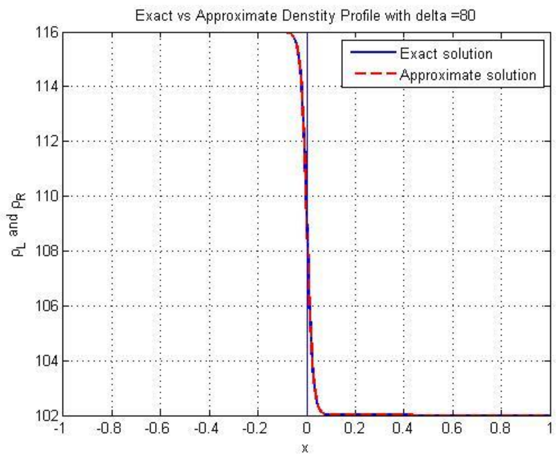
5. Results and Discussion

There is different density under consideration by choosing left and right density, that is, the left density is greater than the right density, so the approximate solution refers to same as left to right and vice versa. In the approximate solution, the initial function is to be considered.

$$\rho_0^\delta = (\rho_L - \rho_R)\frac{e^{-\delta x}}{1+e^{-\delta x}} + \rho_R$$

Where delta is the increasing value. Using different delta values, the exact solution may reach the density considered for the Riemann problem, so for the different values of density for left and right density, a MATLAB code is developed by for the density for left and right density, so by considering the delta values, the approximate solution approaches the exact solution. For solutions with delta, see Fig. 6.





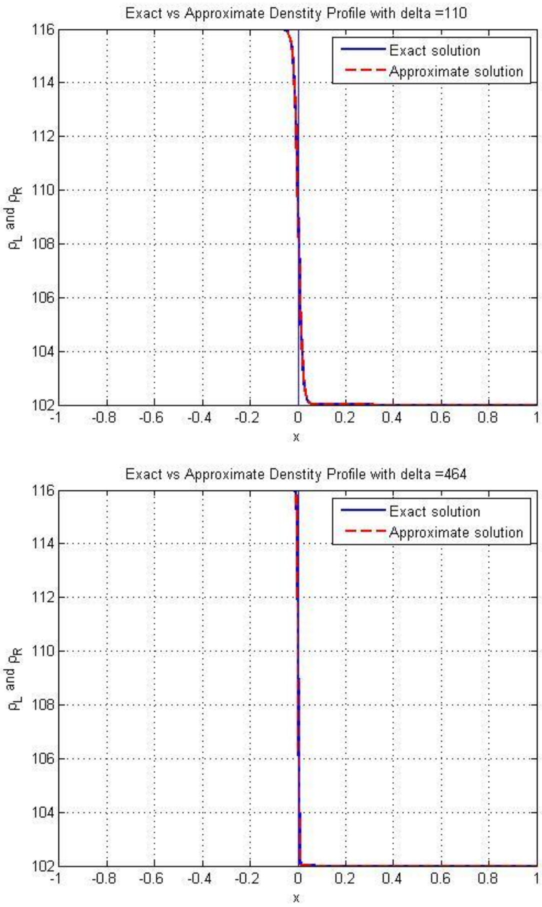
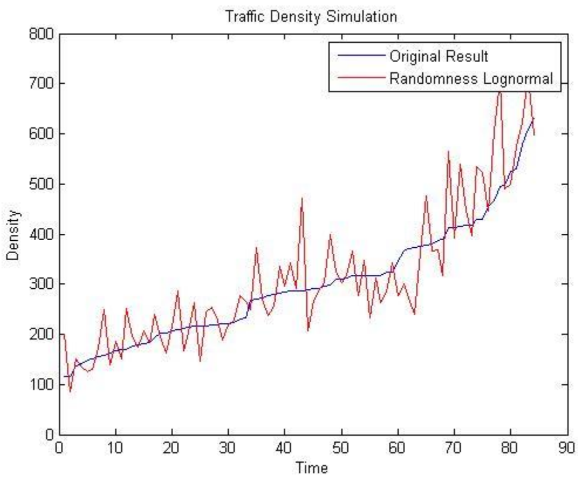


Fig. 6 Exact vs. approximate density profile (developed by the authors)

For the approximate solution, we consider that the left density is 116 greater than the right density 102 the delta is increasing constant number as much greater than the left density and the approximate solution may exit at delta=464, while by using 40, 50, 80, 105, and 110 the approximate solution is not near to the Riemann solver, whereas for the multiply the greater density, hence by increasing the delta value moves to the exact solution of the Riemann solver. The proposed model (Fig. 7) solution with randomness uses normal, lognormal, Weibull, exponential, and gamma distributions with Brownian motion as the lognormal distribution.



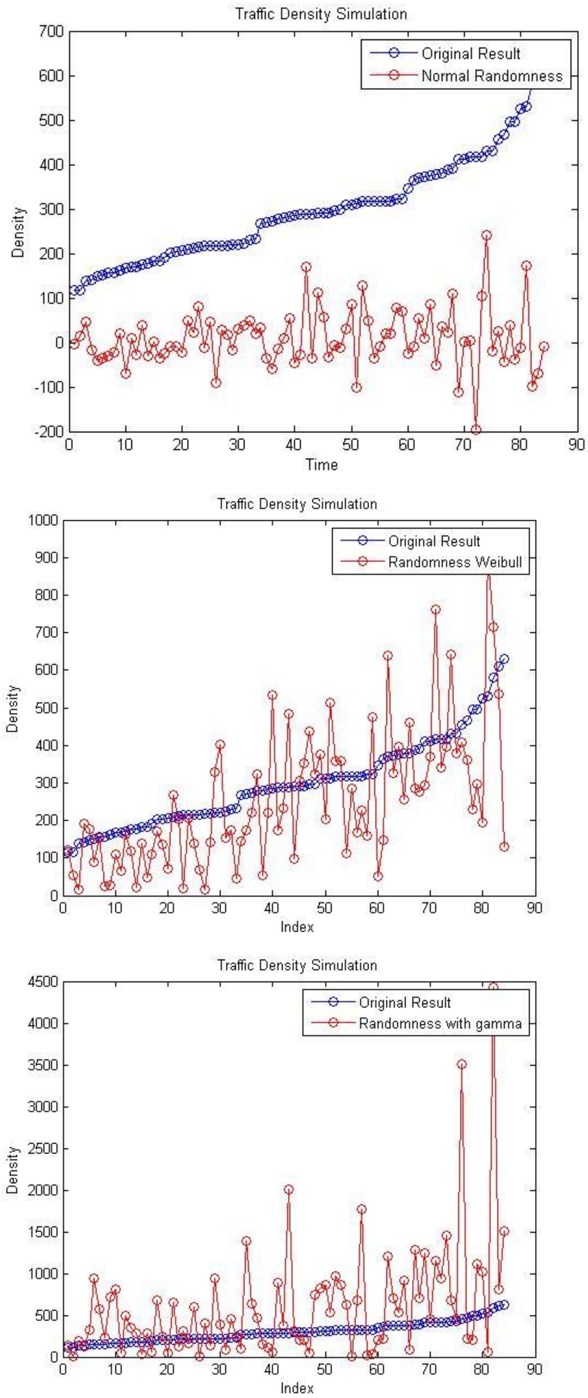
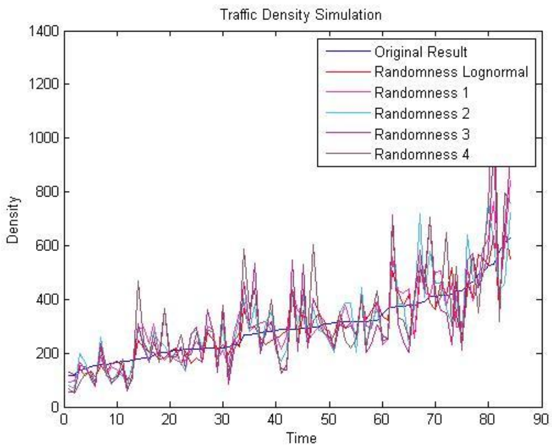


Fig. 7 Time vs. Density with different randomness (developed by the authors)



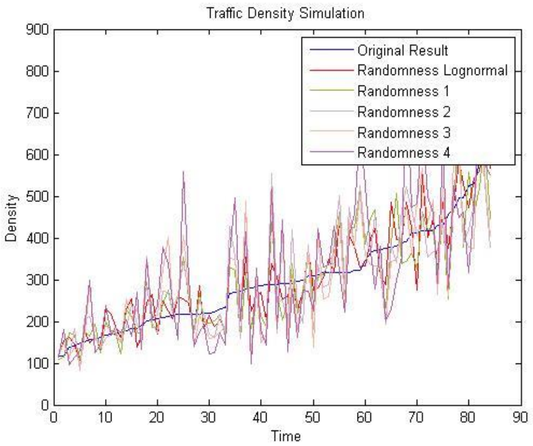


Fig. 8 Randomness/stochastic behavior (developed by the authors)

Fig. 7 shows the solution of the proposed model with different randomness for considering the stochastic behavior of the vehicles flow over path S1, the blue line shows the real data collected over path S1, and the red line describes the solution of the proposed model equation with randomness as Brownian motion using lognormal distribution, which describes the different situations of real traffic data over the road section. As a result, the randomness with lognormal distribution follows the collected data, and there is small fluctuation into the randomness, while other randomness shows a hug fluctuation into the randomness and not following the actual data. Thus, the randomness that follows the best randomness that represents the real-time traffic flows over the paths S1. Similarly, for all other paths, lognormal randomness is the best fit for the proposed stochastic model.

While Fig. 8 shows the stochastic (random) behavior of the model over the initial density, using Brownian motion and randomness, four randomness values were applied while for each randomness, the solution of the model followed the actual density of the model.

5. Conclusion

In this study, a stochastic traffic flow model is proposed with a forcing function using Brownian motion to incorporate the stochastic behavior of traffic flows. This model is an extension of the LWR macroscopic model with a non-homogeneous traffic flow model. The model predicts the randomness of the traffic flow with the collected data using the non-normal distribution of the traffic flow. Distribution techniques demonstrate that stochastic traffic flows are not normally distributed. The stochastic traffic flow follows a lognormal distribution. The proposed model is numerically solved using the Godunov's numerical scheme; this method is suitable for a stochastic traffic flow model due to the nonlinearity in the traffic flow with discontinuity in the traffic flow. The Green-shield model was used as density-speed function for the macroscopic LWR model and the proposed stochastic model. The model achieved high predicted performance with randomness. The proposed method can be used to predict the future state of traffic flow over any section of a road with congestion. The study is based on non-normal distributed traffic data; however, the same study may be carried out with other forms of distributed traffic data, changing the section of road and forcing function, adding some other stochastic factors like signal calibration as discontinuity into the traffic flow, and separating the vehicles types like cars, heavy vehicles, and many others.

References

[1] MOHAN R., and GITAKRISHNAN R. Heterogeneous traffic flow modelling using second-order macroscopic continuum model. *Physics Letters A*, 2017, 381(3): 115-123. https://doi.org/10.1016/j.physleta.2016.10.042

[2] YANG L. *Stochastic traffic flow modeling and optimal congestion pricing*. Dissertation, University of Michigan, 2012 https://deepblue.lib.umich.edu/bitstream/handle/2027.42/94054/youngli_1.pdf?sequence=1

[3] MAHMUD S.M.S., et al. Micro-simulation modelling for traffic safety: A review and potential application to heterogeneous traffic environment. *IATSS Research*, 2019, 43(1): 27-36. https://doi.org/10.1016/j.iatssr.2018.07.002

[4] CUI Z., et al. Traffic graph convolutional recurrent neural network: A deep learning framework for network-scale traffic learning and forecasting. *IEEE Transactions on Intelligent Transportation Systems*, 2019, 21(11): 4883-4894. https://doi.org/ 10.1109/TITS.2019.2950416

[5] WHITHAM G. B. *Linear and nonlinear waves*. John Wiley & Sons, 2011.

[6] DAGANZO C.F. Requiem for second-order fluid approximations of traffic flow. *Transportation Research Part B: Methodological*, 1995, 29(4): 277-286. https://doi.org/10.1016/0191-2615(95)00007-Z

[7] KHABBAZ M. J., WISSAM F. F., and CHADI M. A. A simple free-flow traffic model for vehicular intermittently connected networks. *IEEE Transactions on Intelligent Transportation Systems*, 2012, 13(3): 1312-1326. https://doi.org/10.1109/TITS.2012.2188519

[8] GRAMAGLIA, M., et al. New insights from the analysis of free flow vehicular traffic in highways. *Proceedings of the 2011 IEEE International Symposium on a World of Wireless, Mobile and Multimedia Networks*. IEEE, 2011. https://doi.org/10.1109/WoWMoM.2011.5986384

[9] FAROOQ M. U., MOHAMMAD P., and KHALEEL Ur R. K. A data dissemination model for Cloud enabled VANETs using In-Vehicular resources. *Proceedings of the 2014 International Conference on Computing for Sustainable Global Development*, 2014. https://doi.org/10.1109/IndiaCom.2014.6828180

[10] AKAMATSU T. Cyclic flows, Markov process and stochastic traffic assignment. *Transportation Research Part B: Methodological*, 1996, 30(5): 369-386. https://doi.org/10.1016/0191-2615(96)00003-3

[11] BONZANI I., and MUSSONE L. Stochastic modelling of traffic flow. *Mathematical and Computer Modelling*, 2002, 36(1-2): 109-119.

https://doi.org/10.1016/S0895-7177(02)00107-3

[12] ZHANG T., ROBSON E., and BOUKERCHE A. Design and analysis of stochastic traffic flow models for vehicular clouds. *Ad Hoc Networks*, 2016, (52): 39-49. https://doi.org/10.1016/j.adhoc.2016.07.009

[13] SHAO H., LAM W HK, and TAM M. L. A reliability-based stochastic traffic assignment model for network with multiple user classes under uncertainty in demand. *Networks and Spatial Economics*, 2006, 6: 173-204. https://doi.org/10.1007/s11067-006-9279-6

[14] ALPEROVICH T., and SOPASAKIS A. Stochastic description of traffic flow. *Journal of Statistical Physics*, 2008, 133: 1083-1105. https://doi.org/10.1007/s10955-008-9652-6

[15] NGODUY D. Multiclass first-order traffic model using stochastic fundamental diagrams. *Transportmetrica*, 2011, 7(2): 111-125. https://doi.org/10.1080/18128600903251334

[16] ZHENG F., et al. Traffic state estimation using stochastic Lagrangian dynamics. *Transportation Research Part B: Methodological*, 2018, 115: 143-165. https://doi.org/10.1016/j.trb.2018.07.004

[17] ZHENG F., et al. Analyzing the impact of automated vehicles on uncertainty and stability of the mixed traffic flow. *Transportation Research Part C: Emerging Technologies*, 2020, 112: 203-219. https://doi.org/10.1016/j.trc.2020.01.017

[18] YANG, L. *Stochastic traffic flow modeling and optimal congestion pricing*. Dissertation, University of Michigan, 2012. https://deepblue.lib.umich.edu/bitstream/handle/2027.42/94054/youngli_1.pdf?sequence=1

[19] JABARI S. E., and LIU H.X. A stochastic model of traffic flow: Theoretical foundations. *Transportation Research Part B: Methodological*, 2012, 46(1): 156-174. https://doi.org/10.1016/j.trb.2011.09.006

[20] THONHOFER E., et al. Macroscopic traffic model for large scale urban traffic network design. *Simulation Modelling Practice and Theory*, 2018, 80: 32-49. https://doi.org/10.1016/j.simpat.2017.09.007

[21] SEO T., et al. Traffic state estimation on highway: A comprehensive survey. *Annual Reviews in Control*, 2017, 43: 128-151. https://doi.org/10.1016/j.arcontrol.2017.03.005

[22] WEI C., ASAKURA Y., and IRYO T. The posterior probability distribution of traffic flow: a new scheme for the assignment of stochastic traffic flow. *Transportmetrica A: Transport Science*, 2013, 9(8): 753-771. https://doi.org/10.1080/18128602.2012.661799

[23] ZAMITH M., et al. A new stochastic cellular automata model for traffic flow simulation with drivers' behavior prediction. *Journal of Computational Science*, 2015, 9: 51-56. https://doi.org/10.1016/j.jocs.2015.04.005

[24] REGRAGUI Y., and MOUSSA N. A cellular automata model for urban traffic with multiple roundabouts. *Chinese Journal of Physics*, 2018, 56(3): 1273-1285. https://doi.org/10.1016/j.cjph.2018.02.010

[25] CEMERLIC A. Continuum modeling of the deceleration transient state in stochastic traffic flow. Dissertation, University of Tennessee at Chattanooga, 2015. https://scholar.utc.edu/theses/432/

[26] CALVERT S., et al. Vehicle specific behaviour in macroscopic traffic modelling through stochastic advection invariant. *Transportation Research Procedia*, 2015, 10: 71-81 https://doi.org/10.1016/j.trpro.2015.09.057

[27] DI FEBBRARO A., GIGLIO D., and SACCO N. A deterministic and stochastic Petri net model for traffic-responsive signaling control in urban areas. *IEEE Transactions on Intelligent Transportation Systems*, 2015, 17(2): 510-524. https://doi.org/ 10.1109/TITS.2015.2478602

[28] CHU K.-C., SAIGAL R., and SAITOU K. Stochastic Lagrangian traffic flow modeling and real-time traffic prediction. *Proceedings of the 2016 IEEE International Conference on Automation Science and Engineering (CASE)*. IEEE, 2016. https://doi.org/ 10.1109/COASE.2016.7743383

[29] KENDZIORRA A., WAGNER P., and TOLEDO T. A stochastic car following model. *Transportation Research Procedia*, 2016, 15: 198-207. https://doi.org/10.1016/j.trpro.2016.06.017

[30] SIQUEIRA A. F., et al. Effect of stochastic transition in the fundamental diagram of traffic flow. *Transportation Research Part B: Methodological*, 2016, 87: 1-13. https://doi.org/10.1016/j.trb.2016.02.003.

[31] JABARI S. E., et al. Stochastic Lagrangian modeling of traffic dynamics. *Proceedings of the 97th Annual Meeting Transportation Research Board*, 2018.

[32] SUH J., HEUNGSEOK C., and KYONGSU Y. Stochastic model-predictive control for lane change decision of automated driving vehicles. *IEEE Transactions on Vehicular Technology*, 2018, 67(6): 4771-4782. https://doi.org/10.1109/TVT.2018.2804891

[33] LIN, S., et al. Stochastic link flow model for signalized traffic networks with uncertainty in demand. *IFAC-PapersOnLine*, 2018, 51(9): 458-463. https://doi.org/10.1016/j.ifacol.2018.07.075

[34] NI,; D., HUI K. H., and TAO J. Modeling phase diagrams as stochastic processes with application in vehicular traffic flow. *Applied Mathematical Modelling*, 2018, 53: 106-117. https://doi.org/10.1016/j.apm.2017.08.029

[35] LUO X., LI D., and ZHANG S. Traffic flow prediction during the holidays based on DFT and SVR. *Journal of Sensors*, 2019 (2019) https://doi.org/10.1155/2019/6461450

[36] FAN, T., et al. A dynamically bi-orthogonal solution method for a stochastic Lighthill-Whitham-Richards traffic flow model. *Computer-Aided Civil and Infrastructure Engineering*, 2023, 38(11): 1447-1461. https://doi.org/10.1111/mice.12953

[37] CHENG Q., et al. Analytical formulation for explaining the variations in traffic states: A fundamental diagram modeling perspective with stochastic parameters. *European Journal of Operational Research*, 2024, 312(1): 182-197. https://doi.org/10.1016/j.ejor.2023.07.005

[38] ZHENG S.-T., et al. Impact of stochasticity on traffic flow dynamics in macroscopic continuum models. *Transportation Research Record*, 2020, 2674(10): 690-704. https://doi.org/10.1177/0361198120937704

[39] MARTÍNEZ I., and JIN W.-L. Stochastic LWR model with heterogeneous vehicles: Theory and application for autonomous vehicles. *Transportation Research Procedia*, 2020, 47: 155-162. https://doi.org/10.1016/j.trpro.2020.03.088

参考文:

[1] MOHAN R. 和 GITAKRISHNAN R. 使用二阶宏观连续模型进行非均匀交通流建模。物理快报 A，2017，381(3)：115-123。https://doi.org/10.1016/j.physleta.2016.10.042

[2] YANG L. 随机交通流建模和最优拥堵定价。论文，密歇根大学，2012. https://deepblue.lib.umich.edu/bitstream/handle/2027.42/94054/youngli_1.pdf?sequence=1

[3] MAHMUD S.M.S. 等。 交通安全微观仿真建模：综述及其在非均匀交通环境中的潜在应用。IATSS 研究，2019，43(1)：27-36。https://doi.org/10.1016/j.iatssr.2018.07.002

[4] CUI Z. 等。 交通图卷积循环神经网络：一种用于网络规模交通学习和预测的深度学习框架。IEEE 智能交通系统学报，2019，21(11)：4883-4894。https://doi.org/10.1109/TITS.2019.2950416

[5] WHITHAM G. B. 线性和非线性波。约翰·威利父子公司，2011。

[6] DAGANZO C.F. 交通流二阶流体近似的安魂曲。交通研究 B 部分：方法论，1995，29(4)：277-286。https://doi.org/10.1016/0191-2615(95)00007-Z

[7] KHABBAZ M. J.、WISSAM F. F. 和 CHADI M. A. 车辆间歇性连接网络的简单自由流交通模型。IEEE 智能交通系统学报，2012，13(3)：1312-1326。https://doi.org/10.1109/TITS.2012.2188519

[8] GRAMAGLIA M., 等。 从高速公路自由流车辆交通分析中获得的新见解。2011 年 IEEE 无线、移动和多媒体网络世界国际研讨会论文集。IEEE，2011。https://doi.org/10.1109/WoWMoM.2011.5986384

[9] FAROOQ M. U.、MOHAMMAD P. 和 KHALEEL Ur R. K. 使用车载资源的云支持 VANET 的数据传播模型。2014 年国际计算促进全球可持续发展会议论文集，2014。https://doi.org/10.1109/IndiaCom.2014.6828180

[10] AKAMATSU T. 循环流、马尔可夫过程和随机交通分配。 交通研究 B 部分：方法论，1996，30(5)：369-386。https://doi.org/10.1016/0191-2615(96)00003-3

[11] BONZANI I. 和 MUSSONE L. 交通流随机建模。数学与计算机建模，2002，36(1-2)：109-119。https://doi.org/10.1016/S0895-7177(02)00107-3

[12] ZHANG T.、ROBSON E. 和 BOUKERCHE A. 车辆云随机交通流模型的设计和分析。自组织网络，2016，(52)：39-49。https://doi.org/10.1016/j.adhoc.2016.07.009

[13] SHAO H.、LAM W HK 和 TAM M. L. 需求不确定情况下多用户类别网络的基于可靠性的随机交通分配模型。网络与空间经济学，2006，6：173-204. https://doi.org/10.1007/s11067-006-9279-6

[14] ALPEROVICH T. 和 SOPASAKIS A. 交通流的随机描述。 统计物理学杂志，2008，133：1083-1105。https://doi.org/10.1007/s10955-008-9652-6

[15] NGODUY D. 使用随机基本图的多类一阶交通模型。 运输计量，2011，7（2）：111-125。https://doi.org/10.1080/18128600903251334

[16] ZHENG F., 等。 使用随机拉格朗日动力学进行交通状态估计。 交通研究 B 部分：方法论，2018，115：143-165。https://doi.org/10.1016/j.trb.2018.07.004

[17] ZHENG F., 等。 分析自动驾驶汽车对混合交通流不确定性和稳定性的影响。交通研究 C 部分：新兴技术，2020，112：203-219。https://doi.org/10.1016/j.trc.2020.01.017

[18] YANG, L. 随机交通流建模和最优拥堵定价。论文，密歇根大学，2012。https://deepblue.lib.umich.edu/bitstream/handle/2027.42/94054/youngli_1.pdf?sequence=1

[19] JABARI S. E. 和 LIU H.X. 交通流随机模型：理论基础。 交通研究 B 部分：方法论，2012，46(1)：156-174。https://doi.org/10.1016/j.trb.2011.09.006

[20] THONHOFER E. 等。 大规模城市交通网络设计的宏观交通模型。仿真建模实践与理论，2018，80：32-49。https://doi.org/10.1016/j.simpat.2017.09.007

[21] SEO T. 等。 高速公路交通状态估计：综合调查。年度控制评论，2017，43：128-151。https://doi.org/10.1016/j.arcontrol.2017.03.005

[22] WEI C.、ASAKURA Y. 和 IRYO T. 交通流的后验概率分布：一种新的随机交通流分配方案. 运输计量 A：运输科学，2013，9(8)：753-771。https://doi.org/10.1080/18128602.2012.661799

[23] ZAMITH M. 等。 一种用于交通流模拟和驾驶员行为预测的新型随机细胞自动机模型。计算科学杂志，2015，9：51-56。https://doi.org/10.1016/j.jocs.2015.04.005

[24] REGRAGUI Y. 和 MOUSSA N. 一种用于具有多个环形交叉路口的城市交通细胞自动机模型。 《中国物理学报》，2018，56(3): 1273-1285. https://doi.org/10.1016/j.cjph.2018.02.010

[25] CEMERLIC A. 随机交通流中减速瞬态的连续建模. 田纳西大学查塔努加分校论文，2015. https://scholar.utc.edu/theses/432/

[26] CALVERT S. 等. 通过随机平流不变量进行宏观交通建模中的车辆特定行为.交通研究进展，2015，10：71

-81. https://doi.org/10.1016/j.trpro.2015.09.057

[27] DI FEBBRARO A.、GIGLIO D. 和 SACCO N. 城市交通响应信号控制的确定性和随机 Petri 网模型。IEEE 智能交通系统学报, 2015, 17(2)：510-524。https://doi.org/10.1109/TITS.2015.2478602

[28] CHU K.-C.、SAIGAL R. 和 SAITOU K. 随机拉格朗日交通流建模和实时交通预测。2016. IEEE 自动化科学与工程国际会议 (CASE) 论文集。IEEE, 2016。https://doi.org/10.1109/COASE.2016.7743383

[29] KENDZIORRA A.、WAGNER P. 和 TOLEDO T. 随机跟车模型。《交通研究进展》, 2016, 15：198-207。 https://doi.org/10.1016/j.trpro.2016.06.017

[30] SIQUEIRA A. F. 等。交通流基本图中随机过渡的影响。交通研究 B 部分：方法论, 2016, 87：1-13。https://doi.org/10.1016/j.trb.2016.02.003。

[31] JABARI S. E. 等。 交通动力学的随机拉格朗日建模。 第 97 届交通研究委员会年会论文集, 2018。

[32] SUH J.、HEUNGSEOK C. 和 KYONGSU Y. 自动驾驶汽车车道变换决策的随机模型预测控制。IEEE 车辆技术学报, 2018, 67(6): 4771-4782。https://doi.org/10.1109/TVT.2018.2804891

[33] LIN, S., 等。 需求不确定性条件下信号交通网络的随机链路流模型。IFAC-PapersOnLine, 2018, 51(9): 458-463。https://doi.org/10.1016/j.ifacol.2018.07.075

[34] NI D., HUI K. H., 和 TAO J. NI D., HUI K. H., 和 TAO J. 将相图建模为随机过程并应用于车辆交通流。应用数学建模, 2018, 53: 106-117. https://doi.org/10.1016/j.apm.2017.08.029

[35] LUO X., LI D., 和 ZHANG S.基于DFT和SVR的节假日交通流预测。《传感器杂志》, 2019（2019）https://doi.org/10.1155/2019/6461450

[36] FAN, T., 等。 随机Lighthill-Whitham-Richards交通流模型的动态双正交求解方法。计算机辅助土木和基础设施工程, 2023, 38（11）：1447-1461。https://doi.org/10.1111/mice.12953

[37] CHENG Q., 等。 解释交通状态变化的分析公式：具有随机参数的基本图表建模视角。《欧洲运筹学杂志》, 2024, 312（1）：182-197。https://doi.org/10.1016/j.ejor.2023.07.005

[38] ZHENG S.-T., 等。宏观连续模型中随机性对交n通流动力学的影响。交通研究记录, 2020 , 2674(10)： 690-704。https://doi.org/10.1177/0361198120937704

[39] MARTÍNEZ I. 和 JIN W.-L。异质车辆随机 LWR 模型：自动驾驶汽车的理论与应用。 交通研究进展, 2020, 47：155-162。https://doi.org/10.1016/j.trpro.2020.03.088