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Applications of Multivalued Logic to Set Theory and Calculus

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Abstract: The objectives of this paper are to make set theory a step farther from catastrophic contradictions such as the Russell paradox, which blow away some collections of objects such as the set of all sets, which intuitively should exist since their members exist, and make use of multivalued logic in developing non-standard models in set theory and calculus to make work easier. The set theoretic contradictions are derived through two-valued logic, which ends with sentences of the form “P and not P,” which always have the value “false.” By introducing other truth values, such sentences can end up with a non-false value if both “P” and “not P” are not false. Non-standard objects are obtained through collections of ordinary objects indicating that none of the elements are meant, whereas what is more desirable is indicated. For example, a positive number smaller than all positive real numbers can be indicated by a set of positive real numbers with values ever approaching but never truly attaining “true” as they approach zero. In this way, infinite sets can be indicated by multivalued collections of finite sets, and both infinite and infinitesimal numbers are reintroduced to calculus, improving the scope of integration and differentiation. This paper also presents a new way to accommodate truth values for set theory where the objects can be far more numerous than real numbers between zero and one representing values between “false” and “true.”

Keywords: multivalued logic, the Zermelo-Fraenkel set theory, equipollence classes, truth values, multivalued derivatives and integrals, sums of arbitrary series.

多值逻辑在集合论和微积分中的应用

摘要：本文的目的是使集合论远离诸如罗素悖论之类的灾难性矛盾，它消除了一些对象的集合，例如所有集合的集合，直观上应该存在，因为它们的成员存在，并利用开发集合论和微积分中的非标准模型的多值逻辑，使工作变得更容易。集合论矛盾是通过二值逻辑导出的，它以“磷和非磷”形式的句子结尾，其值始终为“假”。通过引入其他真值，如果“磷”和“不是磷”都不为假，则此类句子最终可以得到非假值。非标准对象是通过普通对象的集合获得的，表明没有任何元素是有意义的，而表明了更需要的元素。例如，小于所有正实数的正数可以由一组正实数表示，这些正实数的值不断接近但在接近零时从未真正达到“真”。这样，无限集可以用有限集的多值集合来表示，无限数和无穷小数都被重新引入微积分中，提高了积分和微分的范围。本文还提出了一种适应集合论真值的新方法，其中对象可以比代表“假”和“真”之间值的零到一之间的实数多得多。

关键词:多值逻辑、策梅洛-

弗兰克尔集合论、等价类、真值、多值导数和积分、任意级数之和。

Introduction

Many-valued logic is logic with truth values that are more than just true or false. For example, let the real numbers in the interval $[0,1]$ be used as its truth values: 0 for false, 1 for true, and the others are intermediate values. If for any sentence P , $v(P)$ denotes the truth value of P , the valuation of composite sentences satisfies the following:

- a. $v(\neg P) = 1 - v(P)$;
- b. $v(P \wedge Q) = \min\{v(P), v(Q)\}$;
- c. $v(P \vee Q) = \max\{v(P), v(Q)\}$;
- d. $v(P \rightarrow Q) = v(\neg P \vee Q) = \max\{v(\neg P), v(Q)\}$
 $= \max\{1 - v(P), v(Q)\}$;
- e. $v(P \leftrightarrow Q) = v((P \rightarrow Q) \wedge (Q \rightarrow P)) =$
 $\min\{\max\{1 - v(P), v(Q)\}, \max\{1 -$
 $v(Q), v(P)\}\}$;
- f. $v(\forall x)Px$
 $= \min\{v(Px) | x \text{ is a relevant term}\}$;
- g. $v(\exists x)Px$
 $= \max\{v(Px) | x \text{ is a relevant term}\}$.

To express that a sentence P has truth value r , we shall write rP with variation to place the left subscript under the outermost connective of P , such as $P \wedge Q$ or $(\forall x)P$. When there is no further notification, we mean $r > 0$.

A set may have elements with diverse nonzero truths of elementhood in it. For example, the set $S = \{a, b, c\}$ where $a_{0.3} \in S$, $b_{0.7} \in S$, and $c_1 \in S$, namely, $c \in S$. In this case, the truth values of elements in S are written as a left subscript for each element, except when the subscript has value 1, when it needs not to be written. Hence, S is written as $\{_{0.3}a, _{0.7}b, c\}$. The truth value of the elementhood of an object a in a set S depends on the object a and the set S . In general, we should write $a_{v(a,S)} \in S$ where $v(a,S)$ denotes the truth value of the sentence $a \in S$. A set is solely determined by the specific truth values of elementhood for its members. Thus, the sets A and B are equal if $(\forall x)(\forall r)(x_r \in A \leftrightarrow x_r \in B)$. A set with all its elements having a truth value 1 is called a crisp set, to borrow terminology from fuzzy set theory. The crisp version of a multivalued set S , if it exists, is called the support of S , written as $Sup(S)$, with capital S to distinguish the notation from the supremum notation in analysis.

If in a set S , the elementhood value of a is true for more than one number, we take its supremum.

One of the applications of the above notions is that if $f: A \rightarrow B$ is a function and $a_r \in A$, $f(a)_r \in B$

because it is equivalent to $_1(f: A \rightarrow B \text{ is a function}) \wedge a_r \in A$, which results in $\min\{r, 1\}(f: A \rightarrow B \text{ is a function} \wedge a \in A)$.

In this paper, the Zermelo-Fraenkel set theory axioms, which are true for crisp sets, are used as guides for considering additional statements that must be fulfilled by multivalued sets. Thereafter, some important theorems are treated similarly. The notion of directed-set objects or constructed objects in [1] is expressible using multivalued logic in calculus, where the real numbers in $[0,1]$ are the truth values. For example, the right-hand parameter at $a \in R$ is defined as a (standardized) variable R_a over real numbers larger than a , and the nested interval $\{(a, x) | x > a\}$ acts as a direction indicator by stipulating that a predicate $Q(R_a)$ is true, which means that there is $x > a$ so that $Q(y)$ is true for any $y \in (a, x)$. Using many-valued logic, this can be altered to an expression $(\forall y)(y_r \in R_a \rightarrow_r Q(y))$, in which we define R_a to be the set of real numbers $x > a$ with elementhood value $0 < r < 1$ but $r \rightarrow 1$ as $x \rightarrow a^+$. Applications such as this are discussed more intensively in this paper.

Constructions of big sets from sequences of finite sets should use another method to produce truth values because sets are way too numerous compared to real numbers. The values chosen in this paper are the sets themselves, or if preferable, their equipollence classes. With the new method, each element is tagged by a truth value that tells us at what stage it is constructed. The larger its elementhood truth value, the earlier its construction occurs. For instance, the set $\{\{_{1,2,3}a, \{_{\{u\}}b, \{_{\{x,y\}}c, \{_{\{v,w\}}d\}\}$ tells us that b is constructed first, c and d the second ones, and a the last, because $\{u\}$ has fewer elements than $\{x, y\}$, $\{x, y\}$ has the same number of elements as $\{v, w\}$, and $\{x, y\}$ has fewer elements than $\{1, 2, 3\}$. The reverse order is used when the negatives of sets are used: $\{-_{\{1,2,3\}}a, -_{\{u\}}b, -_{\{x,y\}}c, -_{\{v,w\}}d\}$. Here, a comes first, c and d are the second ones, and b is the last. This is because for any set X as an index, $-X$ is an abbreviation of $T - X$, where T is a (multivalued) set of all sets. In this case, set subtraction as $T - X$ means the values of the elements of X in T is nullified, namely are made to become zero. Therefore, $\{a, b, c\}$ is considered smaller than $-\{u, v, w\}$ with the assumption that $\{a, b, c\} \cap \{u, v, w\} = \emptyset$, but if we work explicitly with equipollence classes, the latter would present no problem.

The logic discussed in this paper belongs to

consistent logic, in contrast to the interesting recent development of paraconsistent logic, cf. [2]. For a recent application development of multivalued logic, see [3].

1. Basic Theory

This section is devoted to presenting the crisp set background of the last section, which is a little bit more formal and thus more precise.

The Zermelo-Fraenkel set theory, which we shall refer to as ZF, is designed to avoid sets that are too big to remove the Russell paradox produced by the Russell set $\{x|x \notin x\}$. Such a strategy is called a limitation of size policy. There are some other set theories with this policy, such as the von Neumann-Bernays set theory, but we shall restrict our attention to the simplest one.

ZF has one relation symbol, namely the elementhood symbol $\in$, where $a \in B$ is read as “ a is an element of B ” or “ a is a member of B .” Its variant is $B \ni a$ which reads “ B contains a .” A set A is said to be a subset of another set B if $(\forall x)(x \in A \rightarrow x \in B)$. The symbol is $A \subseteq B$ or $B \supseteq A$. If $(\exists x)(x \in B \wedge x \notin A)$ is also true, A is said to be a proper subset of B . The symbol is $A \subset B$ or $B \supset A$. The equality of two sets A and B is met when both $A \subseteq B$ and $B \subseteq A$ are true. The symbol is $A = B$.

ZF is based on nine axioms, built carefully from scratch. The following is presented in [4], but here, we recite them for comparison with the discussion in the subsequent section.

The first is the existence of a set that has no elements, in contrast to any other objects that are not sets where they even have no concept of elementhood.

Axiom 1: Axiom of empty set: There is a set without elements.

$$(\exists x)(\forall y)(y \notin x)$$

The empty set is symbolized as $\emptyset$.

With this axiom, the world has only one set, namely $\emptyset$.

The next is an axiom that states the existence of a set from any object such that the set contains only it as a single element. Such a set is called a singleton.

Axiom 2. Axiom of singletons: For any object, there is a set that has it as its only element.

$$(\forall x)(\exists y)(x \in y \wedge (\forall z)(z \in y \rightarrow z = x))$$

The set y above is symbolized as $\{x\}$.

With the two axioms, there are infinitely many sets that are at least $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and so on.

Now, we proceed to pairs, namely sets each with two elements composed from any two objects that allow duplication, which amounts to producing a singleton.

Axiom 3. Axiom of pairing: Any two objects produce a set that contains only those as its elements.

$$(\forall x)(\forall y)(\exists z)(x \in z \wedge y \in z \wedge (\forall t)(t \in z \rightarrow (t = x \vee t = y)))$$

The set z above is symbolized as $z = \{x, y\}$.

With the three axioms, there are infinitely many sets of zero, one, or two members only, namely $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}\}$, $\{\{\emptyset\}, \{\{\emptyset\}\}\}$, $\{\emptyset, \{\{\emptyset\}\}\}$, $\{\{\emptyset\}, \{\{\{\emptyset\}\}\}\}$, $\{\{\emptyset\}, \{\{\{\emptyset\}\}\}\}$, and so on.

We proceed to the following operation between sets.

Axiom 4. Axiom of union: Given two sets, there is a set whose elements are precisely their own.

$$(\forall x)(\forall y)(\exists z)(t \in z \leftrightarrow (t \in x \vee t \in y))$$

The set z above is symbolized as $z = x \cup y$.

The introduction of the last axiom gives the totality of sets with finite elements.

Next, we deal with the subsets of the sets that we already have.

Axiom 5. Axiom Schema 5 of subsets: Given a set and a condition on elements, there is a set that contains precisely its elements that satisfy the condition.

$$(\forall x)(\exists y)(\forall t)(t \in y \leftrightarrow (t \in x \wedge \varphi(t))),$$

where $\varphi(t)$ is a condition on t , namely a sentence about the variable t .

The symbol for the set y above is $\{t \in x | \varphi(t)\}$.

The reason behind the name “schema” above is that the axiom is in fact infinitely many statements that vary with the condition $\varphi(t)$. The axiom above allows us to extract a subset from a set according to any desired condition. From this axiom, we obtain the notion of set intersection since $x \cap y \stackrel{\text{def}}{=} \{t \in x | t \in y\}$.

Next, we proceed to power set formation. The power set of a set is the set of all its subsets.

Axiom 6. Axiom of power set: Given any set, its power set exists.

$$(\forall x)(\exists y)(\forall t)(t \in y \leftrightarrow t \subseteq x)$$

The symbol of the set y above is $P(x)$.

Now, we are ready to venture into infinity.

Axiom 7. Axiom of infinity: There is an infinite set that contains $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and so on.

$$(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \rightarrow \{y\} \in x))$$

This axiom specifically feeds a single set to the previous axioms, but this feed is special because it is an infinite set, namely a set equinumerous with one of its proper subsets.

Next, we want many infinite sets, not only the set in the above axiom. The idea is to guarantee that the co-domain of any functional condition is a set. A functional condition is a binary predicate $\psi(x, y)$ so that for any objects x, s, t , $\psi(x, s)$ and $\psi(x, t)$ imply $s = t$.

Axiom 8. Axiom schema of replacement: Given a set and functional condition, the co-domain of the condition applied to the set exists.

$$(\forall x)(\exists y)(\forall t)(t \in y \leftrightarrow (\exists s)(s \in x \wedge \psi(s, t)))$$

The notation for the set y above is $\{t | (\exists s)(s \in x \wedge \psi(s, t))\}$ which is too tedious. Maybe the non-standard notation $\psi(x, \cdot)$ is better.

Thus, to prove that the finite von Neumann ordinals form a legitimate infinite set, we use the axiom as follows. For each natural number n , we recursively define $a_0 = \emptyset$ and $a_{n+1} = \{a_n\}$, similarly $b_0 = \emptyset$ and $b_{n+1} = b_n \cup \{b_n\}$. Therefore, for each n , we can define $\psi_{a_k}(s, t)$ to be the condition $s = a_k \wedge t = b_k$ for each $k = 0, 1, \dots, n$. Let u be the set stated in Axiom 7. Let $\psi(s, t)$ be the condition $(\exists x)(x \in u \wedge \psi_x(s, t))$. The desired set is $\psi(u, \cdot)$.

The following axiom is meant to avoid circular sets such as $a \in b \in c \in a$. It also implies that there is no infinite descending sequence $a_1 \ni a_2 \ni \dots$.

Axiom 9. Axiom of regularity: Every nonempty set contains an element that is disjoint from it.

$$(\forall x)(\exists y)(y \in x \wedge x \cap y = \emptyset)$$

There is one more axiom required by most mathematics:

Axiom 10. Axiom of choice: Any family of nonempty sets has a choice set, namely a set whose intersection with any member of the family is a singleton.

$$\begin{aligned} (\forall x) \big((\forall y)(y \in x \rightarrow y \neq \emptyset) \\ \rightarrow (\exists z)(\forall t)(t \in x \\ \rightarrow (\exists u)(z \cap y = \{u\})) \big) \end{aligned}$$

There is a long list of adventures concerning the axiom of choice, notably, the impact of rejecting it, which results in some weird situations, and its acceptance, which leads to the counterintuitive Banach-Tarski “paradox” [5], [6].

In [1], we discuss the definition of a directed set as a family $\{S_t\}_{t>0}$ of sets that are descending or ascending according to the subset relation. The purpose of such a family is to foster the dynamic nature of the definability of a property in the long run. On the one hand, it tells us at what set a property φ is defined at each stage t ; on the other hand, taken as a whole, it may define a new object characterized by such dynamism. For example, if $\varphi(x)$ is the sentence “ $x > \frac{1}{9}$,” $\varphi(t^2)$ is true at all stages $t > \frac{1}{3}$; hence, the new “number” $a = t^2$ has the property $\varphi(a)$ if the directed set employed is $\{(0, t]\}_{t>0}$ since $\varphi(a)$ is true as $t \rightarrow \infty$.

In [7], the following symbols are employed to define the left and right parameters $R_{-\infty} = \{(-\infty, -t)\}_{t>0}$, $R_a = \left\{ \left(a, a + \frac{1}{t} \right) \right\}_{t>0}$, $L_a = \left\{ \left(a - \frac{1}{t}, a \right) \right\}_{t>0}$, and $L_{\infty} = \{(t, \infty)\}_{t>0}$.

To discuss series in the subsequent section, we close this section with a definition of an infinitary operation. In [1], derivation and integration are reformulated using the parameters above. The following definition is used in [1] to handle infinitely many applications of operations as follows.

Definition 1. Infinite operations: Let $\bar{a} = (a_1, a_2, \dots, a_n, \dots)$ be a sequence of elements and $\bar{*} =$

$(*_1, *_2, \dots, *_n, \dots)$ be a sequence of binary operations. Let f be the function $f(t) = \prod_{k=1}^{[t]} a_k \stackrel{\text{def}}{=} \left((\dots ((a_1 *_1 a_2) *_2 a_3) \dots) *_n a_n \right)$, where $[t]$ is the smallest integer k not less than t . Then, $\bar{*}(\bar{a})$ is defined to be $f(L_{\infty})$.

The function f is sometimes written by abuse of notation in sequential notation $f = (f(1), f(2), f(3), \dots)$. It is noted that if there is an occurrence of a unary operation, $F(x)$, one can define $F^*(x, x) \stackrel{\text{def}}{=} F(x)$. If there is an occurrence of a ternary operation, $G(x, y, z)$, one can split G into $G_2(G_1(x, y), z) \stackrel{\text{def}}{=} G(x, y, z)$. Therefore, Definition 1 is much more general than it seems at first sight.

2. Research Method

For the sake of coherence, this section is segmented into two subsections. The first one involves changes in the basic formulations of the ZF set theory. The second one deals with formulations of parameters in multivalued logic, which has developments in calculus and the introduction of valuation outside $[0, 1]$.

The method can be described as follows:

1. Starting from ordinary objects, such as finite sets and real numbers;
2. Multivalued set formations are employed to produce non-standard objects, such as infinite sets, infinite and infinitesimal numbers, and so on;
3. Meanwhile, along with Forming 2, we need to adapt the set theory to satisfy additional axioms specifically involving multivalued sets;
4. Obtaining various new methods and results as a consequence of 2.

2.1. Multivalued Logic and the ZF Set Theory

The ZF axioms mentioned in Section 2 are meant only for crisp sets. Therefore, each of the ZF axioms should be evaluated one by one considering multivalued logic. Additional items to ZF are marked with asterisks.

As indicated in the Introduction, the extension of ZF due to multivalued sets has elementhood subscripts preceding it. The pre-subscript indicates the truth value of the corresponding elementhood symbol. Thus, $x_r \in y$ means that x is an element of y with value r . Hence, $x_1 \in y$ means $x \in y$, while $x_0 \in y$ means $x \notin y$. Also, $x_r \notin y$ means $x_{1-r} \in y$. An element $x_r \in y$ is called an r -element of y .

Definition 2. Multivalued subset:

- a. A set A is said to be a subset of another set B if $(\forall x)(x_r \in A \rightarrow x_s \in B)$ with $r \leq s$.
 $A \subseteq B$ or $B \supseteq A$.
- b. If $(\exists x)(\exists r)(r > 0 \wedge x_r \in B \wedge x \notin A)$ is also true, A is said to be a proper subset of B .
 $A \subset B$ or $B \supset A$.
- c. The equality of two sets A and B is met when

The axiom of infinity is left as it is since it is only a bridge to infinity by affirming the existence of a single

infinite crisp set, from which its multivalued subsets emerge, including those with the same support. For the sake of completeness, we repeat its mention using von Neumann ordinals as follows.

Axiom 7. Axiom of infinity: There is an infinite set that contains $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, and so on.

$$(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \rightarrow y \cup \{y\} \in x))$$

The axiom of replacement now involves two truth values to be reckoned. One comes from elementhood, and the other comes from the condition.

Axiom Schema 8. Addition to the axiom schema of replacement:* Given a set and a functional condition, the co-domain of the condition applied to the set exists.

$$\begin{aligned} &(\forall x)(\exists y)(\forall t) \left((\exists s)(\exists v)(\exists w)(v, w \right. \\ &\quad \left. > 0 \wedge s_v \in x \wedge_w \psi(s, t)) \right. \\ &\quad \left. \rightarrow t_{\min\{v, w\}} \in y \right) \\ &\wedge \left((\exists u)(u > 0 \wedge t_u \in y) \right. \\ &\quad \left. \rightarrow (\exists v)(\exists w)(u \right. \\ &\quad \left. = \min\{v, w\} \wedge s_v \in x \wedge_w \psi(s, t)) \right) \end{aligned}$$

The notation for the set y above is $\left\{ \min\{v_s, w_s\} t \mid (\exists s)(s_v \in x \wedge_w \psi(s, t)) \right\}$, which is even more tedious than before. We may abbreviate this to $\psi(x, *)$, where the asterisk reminds us of its multivalued nature.

The following axiom confines only to crisp sets. So, we just let it be as it is.

Axiom 9. Axiom of regularity: Every nonempty set contains an element that is disjoint from it.

$$\begin{aligned} &(\forall x) \left((\forall r)(\exists y) \left((r > 0 \wedge y \in x) \right. \right. \\ &\quad \left. \left. \rightarrow (\exists z)(z \in x \wedge x \cap z = \emptyset) \right) \right) \end{aligned}$$

One of the multivalued sets that defy this axiom is the Russell set $r = \{x \mid x \notin x\}$. If we let $r_s \in r$, r satisfies the condition $x \notin x$, namely $r \notin r$, where, in this respect, $r_s \notin r$. Thus, $r_{1-s} \in r$. Hence, $s = 1 - s$, namely, $s = 0.5$. So, $r_{0.5} \in r$, whence $r_{0.5} \ni r_{0.5} \ni r_{0.5} \ni r_{0.5} \ni \dots$.

The axiom of choice needs an adaptation with respect to the elementhood value.

Axiom 10. Addition to the axiom of choice:* Any family of nonempty sets has a choice set, namely a set whose intersection with any member of the family is a singleton.

$$\begin{aligned} &(\forall x) \left((\forall y)(y_r \in x \wedge r > 0 \rightarrow y \neq \emptyset) \right. \\ &\quad \left. \rightarrow (\exists z)(\forall t) \left((\exists s)(t_s \in x \wedge s > 0) \right. \right. \\ &\quad \left. \left. \rightarrow (\exists u)(u_w \in z \wedge z \cap t = \{s u\}) \right) \right) \end{aligned}$$

We may notice that the choice set is no longer necessarily a crisp set.

It is an interesting inquiry whether the additional axiom to the Zermelo-Fraenkel set theory can be suppressed to just one axiom schema:

Axiom schema of multivalued comprehension: For

any condition with one variable, there is a multivalued set such that if an object satisfies the condition with a positive truth value, it is a member of the set with a positive truth value not exceeding the former's truth value.

Given any condition $\varphi(x)$ with the variable x , $(\exists M) \left((\forall r) \left((r > 0 \wedge_r \varphi(x)) \rightarrow (\exists s)(0 < s \leq r \wedge x_{s \in M}) \right) \right)$.

The multivalued set M is not unique.

In the construction of large sets with sequences of finite sets, our main concern in [1] is increasing directed sets. The construction of the set of all natural numbers N_0 and its multiple power sets through a sequence of finite sets cannot be reflected merely using real numbers in the interval $[0, 1]$ since the elements involved can be much more numerous than the values in the interval. As a way out, we may use multivalued sets as truth values. Because the totality of all sets is multivalued, it cannot be unique. Therefore, we cannot employ sets themselves directly as values with $\emptyset$ as the number 0 and the totality of all sets as the number 1 and all others are values in between. For each multivalued set X , we should choose a totality of all sets, T , where each element of T has the same value in T . In this way, $T - X$ becomes the removal of X from T . Then, any set S can be written as $T - X$, where X is the complement of S . The choice of T should present no problem if we take T and T' as equivalent when there is a bijective functional condition φ between them, namely $\varphi(t, t')$ with $t \in \text{Sup}(\{v_{(t, T)} t\})$ and $t' \in \text{Sup}(\{v_{(t', T')} t'\})$. If we denote $T - X$ as $-X$, we have sets as truth values.

Definition 4. Negative of a set: sets with equipollence relations as truth values: Let S be a set. Then, $-S$ is the negative of S , which satisfies the following properties:

- $X \subset Y \leftrightarrow -Y \subset -X$;
- $-X \neq \emptyset$ for any set X ;
- $-\emptyset$ represents "true" or 1;
- $\emptyset$ represents "false" or 0;
- $-X \approx -Y$ if $X \approx Y$;
- $\max\{-X, -Y\} = -\min\{X, Y\} = -(X \cap Y)$ and $\min\{X, Y\} = -\max\{X, Y\} = -(X \cup Y)$.

By employing the above seemingly abstract concept of negative sets as the truth value, we can mimic the construction of N_0 as follows:

$N_0 = \{-_0 0, -_1 1, -_2 2, \dots\} = \{-_n n \mid n = 0, 1, 2, \dots\}$, where the values are $0 = \emptyset$ and, in general, $n + 1 = n \cup \{n\}$ if n is known; in other words, $n + 1 = \{0, 1, 2, \dots, n - 1\}$.

Meanwhile, the construction of the power set of N_0 is reflected in the multivalued set: $P(N_0) = \{-_0 P(0), -_1 P(1), -_2 P(2), \dots\} = \{-_n P(n) \mid n = 0, 1, 2, \dots\}$.

The multiple power set is similarly presented:

$$P^k(N_0) = \{-\emptyset P^k(0), -_1 P^k(1), -_2 P^k(2), \dots\} = \{-_n P^k(n) | n = 0, 1, 2, \dots\}.$$

In [1], the infinite-fold power set is defined as $x \in P^\infty(X) \leftrightarrow x = (x_1, x_2, \dots) \wedge (\forall k) (x_k \in P^k(X)) \wedge (\forall l) (x_l \in x_{l+1})$.

For the multivalued version, if $x_l \in r_l$ for any l , $x \in P^\infty(X)$, where $r = \min_l r_l$, namely, $r = -\bigcup_{l=0}^\infty K_l$ if $r_l = -K_l$ for each l .

2.2. Multivalued Logic and Calculus

A multivalued set P can simplify the notion of a directed set presented in Section 2. The issue is now altered to multivalued sets of real numbers.

Definition 2. Kernel of a set: Let $S \subseteq R$ be a multivalued set. Suppose $S = \{r_x | x \in \text{Sup}(S)\}$. The truth peak of S is $T(S) = \sup\{r_x | x \in \text{Sup}(S)\}$. An element $a \in R$ is said to be:

- The left kernel of S if $a = \inf \text{Sup}(S)$ and $\lim_{x \rightarrow a^-} r_x = T(S)$, and $r_x < T(S)$ for all $x \in \text{Sup}(S)$.
- The right kernel of S if $a = \sup \text{Sup}(S)$ and $\lim_{x \rightarrow a^+} r_x = T(S)$, and $r_x < T(S)$ for all $x \in \text{Sup}(S)$.
- The central kernel of S if $\lim_{x \rightarrow a} r_x = T(S)$, and $r_x < T(S)$ for all $x \in \text{Sup}(S)$.
- The truest element of S if $a \in S$ and $(\forall x)(x \in S \rightarrow r_x \leq r_a)$.

If the kernel can be of type a, b, or c, we can just say it as the kernel of S , written as $\ker(S)$.

The concept of a kernel is the multivalued logic equivalent of the concept of expectation in probability theory.

Obviously, $\ker(AB) = \ker(A) \ker(B)$, where $AB = \{ab | a \in A, b \in B\}$.

The multivalued parameters presented below are special cases of left and right kernels for sets of real numbers but with the number 1 as the value of their truest elements.

Definition 3. Multivalued parameters:

- Let $a \in R$. A multivalued set R_a is called a right parameter at a when a is the right kernel of R_a , where $r_x < 1$ for all x and $T(R_a) = 1$.
- Let $a \in R$. A multivalued set L_a is called a left parameter at a when a is the left kernel of R_a , where $r_x < 1$ for all x and $T(R_a) = 1$.
- A multivalued set $R_{-\infty}$ is called a right parameter at $-\infty$ when $-\infty$ is the right kernel of $R_{-\infty}$, where $r_x < 1$ for all x and $T(R_{-\infty}) = 1$.
- A multivalued set L_∞ is called a left parameter at ∞ when ∞ is the left kernel of L_∞ , where $r_x < 1$ for all x and $T(L_\infty) = 1$.

Here, the larger the truth value for elementhood, the closer an element is to the “true value” of the parameter. For example, for any natural number n , there is x so that $x \in L_\infty$ for $r_x \geq 1 - \frac{1}{n}$ since

$\lim_{x \rightarrow \infty} r_x = 1$. Hence, “what” is the number L_∞ is reflected in its values: the bigger a number, the more similar it is to L_∞ . Thus, L_∞ represents ∞ but in a more “tangible” manner since it can be involved in operations.

As in [1], the derivative of f at a is the result of the limiting process of quotients of the form $f'_{x,y}(a) = \frac{f(x)-f(y)}{x-y}$, with $x < a < y$. Instead of taking the limit to obtain $f'(a) = \lim_{x \rightarrow y} f'_{x,y}(a)$, employing R_0 , which represents a tangible positive infinitesimal, we define:

Definition 5. Multivalued derivative: Let f be a function, and a is in the interior of its domain. The multivalued derivative of f at a is

$$f^*(a) = \{v_{(d, R_0)} f'_{x,y}(a) | |x - y| = d, d > 0\} \dots \quad (1)$$

where $f'_{x,y}(a) = \frac{f(x)-f(y)}{x-y}$ for $x < a < y$, and $v(d, R_0)$ means the elementhood value of d in R_0 .

In short, the derivative of f at a is a set with elements of the form $\frac{f(x)-f(y)}{x-y}$, where $x < a < y$ with elementhood value $v(|x - y|, R_0)$, namely, the value is increasingly higher when it approaches zero.

Let us now evaluate some of the main theorems of differential calculus, the details of which are found in [9] or [10] for reference to the ordinary versions.

Theorem 1. The chain rule: Let $f: A \rightarrow B$ and $g: B \rightarrow R$ be functions, where $A, B \subseteq R$. Let $a \in A$. Then, $\ker(g \circ f)^*(a) = \ker(g^*(f(a)))f^*(a)$.

Proof: Suppose $w = \ker(g \circ f)^*(a)$. Then, $w = \lim_{u \rightarrow 1^-, d_u = v(u, (g \circ f)^*(a))} u$. However, the typical element

$$\text{of the support of } (g \circ f)^*(a) \text{ is } w = \frac{(g \circ f)(x) - (g \circ f)(y)}{x - y} = \frac{g(f(x)) - g(f(y))}{x - y}$$

$$= \frac{g(f(x)) - g(f(y))}{f(x) - f(y)} \cdot \frac{f(x) - f(y)}{x - y} \in \text{Sup}(g^*(f(a)))\text{Sup}(f^*(a)).$$

$$\text{Now, } v(w, (g \circ f)^*(a)) = v(x - y, R_0),$$

$$v\left(\frac{g(f(x)) - g(f(y))}{f(x) - f(y)}, g^*(f(a))\right)$$

$$= v(|f(x) - f(y)|, R_0), \quad \text{and}$$

$$v\left(\frac{f(x) - f(y)}{x - y}, f^*(a)\right) = v(|x - y|, R_0).$$

So, $v(w, (g \circ f)^*(a)) = v(|f(x) - f(y)|, R_0)v(|x - y|, R_0)$. Hence, if $v(w, (g \circ f)^*(a)) \rightarrow 1^-$, $v(|f(x) - f(y)|, R_0) \rightarrow 1^-$, and $v(|x - y|, R_0) \rightarrow 1^-$.

Thus, $w = \ker(g^*(f(a))) \ker f^*(a) = \ker g^*(f(a))f^*(a)$.

On the contrary, if $p = \ker(g^*(f(a))f^*(a))$, $p = \ker(g^*(f(a))) \ker(f^*(a)) =$

$$\begin{aligned}
& \left(\lim_{d(|f(x_1)-f(y_1)|, R_0) \rightarrow 1^-} \frac{g(f(x_1)) - g(f(y_1))}{f(x_1) - f(y_1)} \right) \\
& \left(\lim_{d(|x_2 - y_2|, R_0) \rightarrow 1^-} \frac{f(x_1) - f(y_2)}{x_2 - y_2} \right) \\
& = \left(\lim_{d(|f(x) - f(y)|, R_0) \rightarrow 1^-} \frac{g(f(x)) - g(f(y))}{f(x) - f(y)} \right) \\
& \left(\lim_{d(|x - y|, R_0) \rightarrow 1^-} \frac{f(x) - f(y)}{x - y} \right) \\
& = \lim_{d(|x - y|, R_0) \rightarrow 1^-} \left(\frac{g(f(x)) - g(f(y))}{f(x) - f(y)} \right) \left(\frac{f(x) - f(y)}{x - y} \right) \\
& = \lim_{d(|x - y|, R_0) \rightarrow 1^-} \frac{g(f(x)) - g(f(y))}{x - y} \\
& = \ker(g \circ f)^*(a).
\end{aligned}$$

Theorem 2. The product rule: Let $f: A \rightarrow R$ and $g: A \rightarrow R$ be functions, where $A \subseteq R$. Let $a \in A$. Then, $\ker((fg)^*(a)) = \ker(f^*(a)g(a) + f(a)g^*(a))$.

Proof: Let $w = \ker((fg)^*(a))$. Then, $w = \lim_{d_u \rightarrow 1^-, d_u = v(u, (fg)^*(a))} u$. However, the typical element of the support of $(fg)^*(a)$ is $w = \frac{(fg)(x) - (fg)(y)}{x - y} = \frac{f(x)g(x) - f(y)g(y)}{x - y} = \frac{f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)}{x - y}$

$$= \frac{f(x)g(x) - f(y)g(x)}{x - y} + \frac{f(y)g(x) - f(y)g(y)}{x - y}$$

$$\in \text{Sup}((f^*(a)g(a)) + \text{Sup}(f(a)g^*(a))).$$

Now, $v(w, (fg)^*(a)) = v(|x - y|, R_0) = v\left(\frac{f(x)g(x) - f(y)g(y)}{x - y}, g^*(f(a))\right) = v\left(\frac{f(y)g(x) - f(y)g(y)}{x - y}, f^*(a)\right)$.

Therefore, if $v(w, (fg)^*(a)) \rightarrow 1^-$, $v\left(\frac{f(x)g(x) - f(y)g(y)}{x - y}, g^*(f(a))\right) \rightarrow 1^-$, and $v\left(\frac{f(y)g(x) - f(y)g(y)}{x - y}, f^*(a)\right) \rightarrow 1^-$.

Thus, $w = \ker(f^*(a)g(a) + \ker(f(a)g^*(a)) = \ker(f^*(a)g(a) + f(a)g^*(a))$.

On the contrary, if $p = \ker(f^*(a)g(a) + f(a)g^*(a))$, $p = \ker(f^*(a)g(a) + f(a)g^*(a)) + \ker(f(a)g^*(a)) =$

$$\begin{aligned}
& \left(\lim_{d(|x_1 - y_1|, R_0) \rightarrow 1^-} \frac{f(x_1)g(x_1) - f(y_1)g(x_1)}{x_1 - y_1} \right) + \\
& \left(\lim_{d(|x_2 - y_2|, R_0) \rightarrow 1^-} \frac{f(x_2)g(x_2) - f(y_2)g(y_2)}{x_2 - y_2} \right) = \\
& \left(\lim_{d(|x - y|, R_0) \rightarrow 1^-} \frac{f(x)g(x) - f(y)g(x)}{f(x) - f(y)} \right) + \\
& \left(\lim_{d(|x - y|, R_0) \rightarrow 1^-} \frac{f(x)g(x) - f(x)g(y)}{f(x) - f(y)} \right) = \\
& \lim_{d(|x - y|, R_0) \rightarrow 1^-} \left(\frac{f(x)g(x) - f(y)g(x)}{f(x) - f(y)} \right) +
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{f(x)g(x) - f(x)g(y)}{f(x) - f(y)} \right) = \\
& \lim_{d(|x - y|, R_0) \rightarrow 1^-} \frac{f(x)g(x) - f(x)g(y)}{x - y} = \ker(fg)^*(a). \square
\end{aligned}$$

The conformity of a multivalued entity with another can only mean the equality of the kernels, as in the following theorem.

Theorem 3. Conformity with the ordinary derivative: Suppose f is a function differentiable in the ordinary sense at $a \in R$. Then, $\ker(f^*(a)) = f'(a)$.

Therefore, the multivalued derivative of f at a is essentially the same as the ordinary derivative since it is its kernel.

Proof: Let f be a function differentiable at a point a . Given $\varepsilon > 0$, there is $\delta > 0$ so that $\left| \frac{f(x) - f(y)}{x - y} - f'(a) \right| < \varepsilon$, provided that $|x - y| < \delta$. Therefore, $f^*(a)$ has kernel $f'(a)$ since if $\delta = R_0$, elementhood value of ${}_df'_{x,y}(a)$ increases as $\delta \rightarrow 0^+$, along with $f'_\delta(a) \rightarrow f'(a)$.

The definitions of partition and its mesh are standard ways to introduce the Riemann integral. Equally central to the idea is the concept of a choice function for each subinterval resulting from the partition.

Definition 6. Partition and its mesh and sum: Let (a, b) be a real interval, where $a = -\infty$ or $b = \infty$ are also allowed.

a. A partition of I is a finite sequence $P = (x_1, x_2, \dots, x_n)$ so that $a = x_1 < x_2 < \dots < x_n = b$.

b. The mesh of P is the number $\max\{x_k - x_{k-1} | k = 2, 3, \dots, n\}$.

The mesh of the partition P is written as $|P|$.

c. The sum of f over the interval $[a, b]$ determined by the partition P is defined as $S(f, P, d, \sigma) = \sum_{k=1}^n f(x_k^*)(x_{k+1} - x_k)$ for $x_i \in P$, where P is a partition of $[a, b]$ with $|P| = d$, and σ is called a choice function for P , denoted by $\sigma \in c(P)$, and fulfills $\sigma([x_k, x_{k+1}]) = x_k^* \in [x_k, x_{k+1}]$ for $k = 1, 2, \dots, n - 1$.

The multivalued set version of the Riemann integral is as follows:

Definition 7. Multivalued integral: Let f be a function defined on the closed interval $[a, b]$. Then, the multivalued Riemann integral is given by

$$\int_a^b * f(x) dx \stackrel{\text{def}}{=} \left\{ v(d, R_0) S(f, P, \sigma) \mid \sigma \in c(P), |P| = d, d > 0 \right\} \dots (2)$$

In the following, conformity is measured by the identity of the kernels.

Theorem 4. Conformity with the ordinary integral: If f is a function integrable in the ordinary sense,

$$\ker \left(\int_a^b * f(x) dx \right) = \int_a^b f(x) dx.$$

Proof: Suppose f is a function with $\int_a^b f(x) dx$

exists. Given $\varepsilon > 0$, there is $\delta > 0$ so that for any partition P with mesh less than δ and any σ , $|S(f, P, \sigma) - \int_a^b f(x)dx| < \varepsilon$. Then, any member $q \in (2)$ with $r > v(\delta, R_0)$ satisfies $|q - \int_a^b f(x)dx| < \varepsilon$ because the corresponding mesh must be smaller than δ . Since ε is arbitrary, the kernel of (2) is indeed $\int_a^b f(x)dx$.

The following theorem involves a multivalued antiderivative.

Theorem 5. Fundamental theorem of calculus: Let $f: [a, b] \rightarrow R$ be a function and $F(x) = \int_a^x f(t)dt$ for $x \in [a, b]$. Then, $\ker \int_a^b f(x)dx = \ker(F(b) - F(a))$.

Proof: Let P be a partition of $[a, b]$ with mesh $|P|$ and $\sigma: P \rightarrow [a, b]$ be a choice function for P . Then, $S(f, P, |P|, \sigma) \in \int_a^b f(x)dx$. Now, $S(f, P, |P|, \sigma) = \sum_{i=1}^n f(x_i^*)(x_{i+1} - x_i) \xrightarrow{|P| \rightarrow 0^+} \sum_{i=0}^n (F(x_{i+1}) - F(x_i)) = F(x_{n+1}) - F(x_0) = F(b) - F(a)$.

Because this is true for $|P|$ small enough, namely $v(|P|, R_0)$ close enough to 1, the theorem follows.

The following essentially states that multivalued integrals and derivatives are inverses.

Theorem 6. Multivalued integrals as multivalued anti-derivatives: Let $f: [a, b] \rightarrow R$ be a function. Then, $\ker(f(a) + \int_a^x f^*(t)dt) = f(x)$ and $\ker(\int_a^x f(t)dt)^* = f(x)$.

Proof: Let $f: I \rightarrow R$ be functions, where $I = [a, b]$. Then, $\ker \int_a^x f^*(t)dt = \ker \int_a^x \ker f^*(t)dt \xrightarrow{|u-v| \rightarrow 0^+} \ker \int_a^x \frac{f(u) - f(v)}{u - v} dt \xrightarrow{|P| \rightarrow 0^+} \sum_{i=1}^n \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} (x_{i+1} - x_i) = f(x) - f(a)$.

Thus, $\ker(f(a) + \int_a^x f^*(t)dt) = \ker(f(a) + f(x) - f(a)) = f(x)$.

For the second part, $\ker(\int_a^x f(t)dt)^* \xrightarrow{|u-v| \rightarrow 0^+} \ker \frac{\int_a^u f(t)dt - \int_a^v f(t)dt}{u - v} \xrightarrow{|P| \rightarrow 0^+} \frac{\sum_{i=1}^{n+1} f(x_{i+1}^*)(x_{i+1} - x_i) - \sum_{i=1}^n f(x_{i+1}^*)(x_{i+1} - x_i)}{u - v} = \frac{f(x_{n+1}^*)(u - v)}{u - v} = f(v^*) = f(x)$,

where we assume that $[a, v] \subseteq [a, u]$ and the partition for $[a, v]$ is $P = (a = x_1, x_2, \dots, x_n = v, x_{n+1} = u)$. $\square$

Integration by parts regains its truth through kernels.

Theorem 7. Integration by parts: Let $f, g: I \rightarrow R$ be functions, where $I = [a, b]$. Then, $\ker(\int_a^b f(x)g^*(x)dx) = \ker((f(x)g(x))_{x=a}^{x=b} - \int_a^b f^*(x)g(x)dx)$.

Proof: By the product rule, $\ker(fg)^*(x) = \ker(f^*g)(x) + \ker(fg^*)(x)$ for each $x \in [a, b]$. Hence, $\ker(\int_a^b (fg)^*(x)dx) = \ker(\int_a^b \ker(fg)^*(x)dx) = \ker(\int_a^b (\ker(f^*g)(x) + \ker(fg^*)(x))dx) = \ker(\int_a^b \ker(f^*g)(x)dx) + \ker(\int_a^b \ker(fg^*)(x)dx) = \ker(\int_a^b (f^*g)(x)dx) + \ker(\int_a^b (fg^*)(x)dx)$.

So, $\ker(\int_a^b (fg^*)(x)dx) = \ker(\int_a^b (fg)^*(x)dx) - \ker(\int_a^b (f^*g)(x)dx)$.

By the first part of Theorem 6, $\ker(\int_a^b (fg)^*(x)dx) = fg(b) - fg(a)$, so the theorem follows.

Some other types of integrals, such as those discussed in [11], can be treated in the same way we do here. Another concept is the inteduct that is related to integrals [12].

We conclude this section with an adaptation of Definition 1 about a type of infinitary operation.

Definition 8. Multivalued version infinite operations: Let $\bar{a} = (a_1, a_2, \dots, a_n, \dots)$ be a sequence of elements and $\bar{*} = (*_1, *_2, \dots, *_n, \dots)$ be a sequence of binary operations. Let $s_n = \prod_{k=1}^n a_k \stackrel{\text{def}}{=} ((\dots((a_1 *_1 a_2) *_2 a_2) *_3 \dots) *_n a_n)$.

Then, $\bar{*}(\bar{a})$ is defined as the set $\{v_{(n, L_\infty)} s_n | n = 1, 2, 3, \dots\}$.

3. Results and Discussion

This section is devoted to the consequences, examples, and applications of what is developed in the last section. Our results on numbers are comparable to surreal numbers [13].

3.1. Five Theorems of the ZF Set Theory Adapted to Multivalued Nature

We now take a glimpse into five big theorems of set theory to determine the impact of multivalued logic on them.

In the following, what is meant by well ordering on a set is that every non-empty subset has a first element.

Theorem 8. The well-ordering theorem: Every multivalued set can be well ordered.

Proof: Let S be a multivalued set. Let $W = (Sup(S), \leq)$ be well ordering of $Sup(S)$. Returning the

elementhood values of S , we obtain well ordering $(S, \leq)$ of S .

The concepts of equipollence or elementhood equinumerosity among sets and thus also cardinality are affected by the multivalence of truth values. First, we must alter the definition of equipollence.

Definition 9. Multivalued equipollence: Let A and B be sets.

a. A is said to be equipollent to B if there is a bijection $f: A \rightarrow B$ such that $(\forall x)(x \in A \leftrightarrow f(x) \in B)$.

$A \approx B$.

b. Suppose α is the cardinality of $\text{Sup}(A)$. The cardinality of A is the set α^* with $\text{Sup}(\alpha^*) = \alpha$ so that there is a bijection $f: A \rightarrow \alpha^*$, where $(\forall x)(x \in A \leftrightarrow f(x) \in \alpha^*)$. Of course, there are many multivalued sets that can be cardinalities of A , but they are deemed equivalent.

c. A is said to be subequipollent to B if there is an injection $g: A \rightarrow B$ so that A and $g(A)$ are equipollent. $A \leq B$.

d. If $A \leq B$, but $A \not\approx B$, A is said to be strictly subequipollent to B . The symbol is $A < B$.

Theorem 9. The multivalued Schroeder-Bernstein theorem: Let A and B be multivalued sets. If $A \leq B$ and $B \leq A$, $A \approx B$.

Proof: Suppose $A \leq B$ and $B \leq A$. The proof follows the original proof of the Schroeder-Bernstein theorem but now involves injections with elementhood truth values. This would present no problem if we use sets as the values, as in Definition 6, instead of real numbers since it allows infinitely small quantities. $\square$

The trichotomy theorem states that for any crisp sets A and B , there are only three alternatives, namely $A \leq B$, $A = B$, or $B \leq A$. However, for multivalued sets in general, the theorem fails.

Theorem 10. Modified trichotomy theorem: Let A and B be multivalued sets. Then, $\text{Sup}(A)$ and $\text{Sup}(B)$ satisfy the trichotomy alternatives, but A and B may be incomparable.

Proof: Let $A = \{_{0,3}u, _{0,7}v\}$ and $B = \{_{0,7}u, _{0,3}v\}$. Then, $A \not\leq B$, and $B \not\leq A$, although $\text{Sup}(A) = \text{Sup}(B)$. B and $C = A \cup \{_{0,5}w\}$ give incomparability of B and C , along with $\text{Sup}(B) \leq \text{Sup}(C)$.

For multivalued sets, the usual argument for Cantor's Theorem cannot be sustained. We know that there is a crisp set, and its crisp power set is not equipollent. If a multivalued set S is equipollent to its multivalued power set via a function $f: S \rightarrow P(S)$, suppose $M \in P(S)$, where $M = \{x | x \notin f(x)\}$. Then, there is $s > 0$ so that $m_s \in S$ is the element such that $f(m_s) = M$. Then, for some r , $m_r \in M \leftrightarrow m_r \notin M \leftrightarrow m_{1-r} \in M$ so that $r = \frac{1}{2}$. So, s should be $\frac{1}{2}$ as well. Therefore, the usual argument fails. Thus, we need another argument for it.

For multivalued sets, Cantor's theorem still holds.

Theorem 11. Cantor's theorem: For any multivalued set S , $S \not\approx P(S)$.

Proof: Let $0 < r < 1$ and $X = \{_ra\}$. Then, $P(X) = \{X_s | s \in (0, r]\}$, where $X_s = \{\emptyset\} \cup \{_sa\}$ for any $s \in (0, r]$. Hence, $P(X) \approx (0, r]$. Now, since $(0, r] \approx (0, 1] \approx R$ for any $r > 0$, the totality of subsets of any set S with support C has the cardinality of $C \times R$. Hence, $P(S)$ has the cardinality of $P(\text{Sup}(C)) \times R$. Since $S \approx C < \text{Sup}(C) < P(\text{Sup}(C))$, $S \leq C \times R < P(\text{Sup}(C)) \times R \approx P(S)$.

Ordinal numbers are expanded to include multivalued sets. Thus, if α is an ordinal number, a multivalued version of α is a set α^* so that $(\forall x)(x \in \alpha \leftrightarrow (\exists r)(r > 0 \wedge y \in \alpha^* \wedge y = x))$, namely, the multivalued version of an ordinal α is a set with support α .

Ordinal addition and limiting are modified as follows.

Definition 10. Ordinal summation:

a. If α is an ordinal and γ is a multivalued successor ordinal, $\gamma = \delta + 1$, and $\alpha + \gamma = (\alpha + \delta) \cup \{_ra\}$, where $r = \sup\{s | (\exists \beta)(\beta \in \alpha + \delta)\}$.

b. If α is an ordinal and γ is a multivalued limit ordinal, $\alpha + \gamma = \bigcup_{\varepsilon \in \gamma} \alpha + \varepsilon$.

For axiomatic classical ordinal theory, see [14]. For a relatively recent finding on ordinals, see [15].

In addition to the well-known transfinite induction for crisp sets, it must be modified as there are now infinitely many additional ordinals.

Theorem 12. Truth value of transfinite induction: Let set equipollence classes be employed for the truth values as in Definition 3. Let $\varphi(x)$ be a predicate on ordinals. Suppose:

a. ${}_r\varphi(0)$ holds for some $r > 0$;

b. For any $s > 0$, if ${}_s\varphi(\alpha)$ holds, ${}_t\varphi(\alpha + 1)$, where $t = \min\{s, u\}$, in which $\alpha_u \in \alpha + 1$;

c. If for any $\alpha_u \in \beta$, ${}_s\varphi(\alpha)$ holds for some $u(\alpha), s(\alpha) > 0$, ${}_t\varphi(\beta)$, where $t = \sup\{\min\{s(\alpha), u(\alpha)\} | \alpha \in \text{Sup}(\beta)\}$.

Then, ${}_{u(\alpha)}\varphi({}_s\varphi(\alpha))$, namely $\min\{u(\alpha), s(\alpha)\}\varphi(\alpha)$, is true for all ordinals.

Proof: The proof of this theorem is similar to the proof of the original version, except that now elementhood values should be considered.

Because the truth values are set equipollence classes, we can be sure that none of the truth values would reduce to zero if none of them are zero. Thus, the theorem follows.

3.2. Limits of Functions, Multivalued Derivatives and Integrals, and Sums of Infinite Series

We now turn to the application of multivalued calculus by considering some examples.

Example 1: At $a > 0$, the function $f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 0, & \text{otherwise,} \end{cases}$ obviously has no derivative in the

ordinary sense. For $d > 0, x < a < y$ with $x - y = d$,

$$\frac{f(x) - f(y)}{x - y}$$

$$= \begin{cases} 0, & \text{if } x \text{ and } y \text{ are both irrational,} \\ \frac{x - y}{d} = 1, & \text{if } x \text{ and } y \text{ are both rational,} \\ \frac{x}{d}, & \text{if } x \text{ is rational and } y \text{ irrational,} \\ -\frac{y}{d}, & \text{if } x \text{ is irrational and } y \text{ rational.} \end{cases}$$

Elements of $f^*(a)$ with large elementhood values, namely $|d| < \frac{a}{2}$ are among $\left\{v(d, R_0)0, v(d, R_0)1, v(d, R_0)\left(\frac{x}{d}\right), v(d, R_0)\left(-\frac{y}{d}\right)\right\}$ where d is ever decreasing as the truth value increases. Hence, $f^*(0)$ has three parts: the 0 part gives 0, the 1 part gives 1, the $\frac{x}{d}$ part gives aL_∞ , and the $-\frac{y}{d}$ gives $-aL_\infty$. Thus, since $a > 0$, $f^*(0)$ is a new entity that glues together the three parts, namely $L_\infty \cup 0 \cup 1 \cup -L_\infty$. We may interpret that the slope of the function has three components, one being horizontal, one being like the derivative of $g(x) = x$, another being vertically upward, and the last one being vertically downward.

In contrast to the ordinary Riemann integral, the multivalued Riemann integral always gives values. Similarly to the multivalued derivative, it can be employed to study different cases of failure handled in [7].

Example 2: Let f be the function of Example 1, where $a = 1$. Then, f has no Riemann integral in the ordinary sense. Let n be a natural number. Let $I = [1, 2]$ and P be the corresponding partition of the arbitrary mesh.

$$\sum_{k=1}^{n-1} f(x_k^*)(x_{k+1} - x_k) \in \left[0, 1\frac{1}{2}\right] \dots \quad (3)$$

because for any $c \in [0, 1]$, if we let $x_1 = 1 < x_2 < \dots < x_k = 1 + c < x_{k+1} < \dots < x_n = b$, x_j^* is rational for $j = 1, \dots, k$, and x_l^* is irrational for any other l 's, $\sum_{k=1}^{n-1} f(x_k^*)(x_{k+1} - x_k) \rightarrow \int_1^{1+c} x dx = \frac{1}{2}(1+c)^2 - \frac{1}{2} = c + \frac{c^2}{2}$ as $|P| \rightarrow 0$.

Thus, $c + \frac{c^2}{2} v(|P|, R_0) \in \int_1^2 f(x) dx$. Since it is arbitrarily close to 0, $c + \frac{c^2}{2} \in \int_1^2 f(x) dx$. Furthermore, since c is arbitrary, (3) follows. The interpretation of this is that the integral is a “number” that encompasses the whole interval $\left[0, 1\frac{1}{2}\right]$.

Example 4: The so-called parametric integration in [7] is also an example that may be undefined in the ordinary sense. For example, $\int_0^\infty \cos x dx$ is undefined, but $\int_0^\infty * \cos x dx$ is a “number” with the truest values obtained by $\sin L_\infty - 1$, namely the interval $[-2, 0]$.

Considering functions of the form $f: R^* \rightarrow R^*$ with $R^* \stackrel{\text{def}}{=} \{R_{-\infty}\} \cup \{L_\infty\} \cup \bigcup_{a \in R} \{R_{-a}, a, L_a\}$, there are

some interesting facts, such as the following, which is proven in [1] using a different approach.

Theorem 13. Value of the integral of infinitesimal extension: Let $f: I \rightarrow R$ be a real function, where I is an interval $[a, b]$. For any $K \in R \cup \{\infty, -\infty\}$, there is an expansion $f^*: R^* \rightarrow R^*$ of f so that $\int_a^b * f^*(x) dx = K$ regardless of the value of $\int_a^b f(x) dx$.

Proof: Suppose $c \in I$, $M = \int_a^b f(x) dx$, and
$$f^*(x) = \begin{cases} f(x), & \text{if } x \in I - (c, R_c), \\ \frac{-(M-K^*)}{R_c - c}, & \text{if } x \in (c, R_c), \text{ with } K^* = \\ 0, & \text{elsewhere,} \end{cases}$$

$$\begin{cases} K, & \text{if } -\infty < K < \infty, \\ L_\infty, & \text{if } K = \infty, \\ R_{-\infty}, & \text{if } K = -\infty. \end{cases}$$

We calculate: $\int_a^b * f^*(x) dx = \int_a^c f(x) dx + \int_c^{R_c} * f^*(x) dx + \int_{R_c}^b f(x) dx$

$$= \int_a^c f(x) dx + \int_c^{R_c} \left(\frac{-(M-K^*)}{R_c - c} \right) dx + \int_{R_c}^b f(x) dx$$

$$= \int_a^b f(x) dx + \int_c^{R_c} \left(\frac{-(M-K^*)}{R_c - c} \right) dx$$

$$= \int_a^b f(x) dx + \int_c^{R_c} * f^*(x) dx + \int_c^{R_c} \left(\frac{-(M-K^*)}{R_c - c} \right) dx.$$

$$\begin{aligned} \text{So, } \int_a^b * f^*(x) dx &= \left(\int_a^c f(x) dx + \ker \int_{R_c}^b * f(x) dx \right) \\ &\quad + \ker \int_c^{R_c} * \left(\frac{-(M-K^*)}{R_c - c} \right) dx \\ &= \left(\int_a^b f(x) dx \right) + \ker \int_c^{R_c} * \left(\frac{-(M-K^*)}{R_c - c} \right) dx \\ &= M - (M - K^*) = K^* = K. \end{aligned}$$

Closely related to calculus is the issue of the sums of series. Definition 2 gives functional equivalents of sums of convergent series. The following are two examples.

Example 5: Convergent series:

$$\begin{aligned} \text{A. } \sum_{n=1}^\infty (-1)^{n+1} \frac{1}{n} &= \\ \left\{ v(n, L_\infty) \left(\sum_{k=1}^n (-1)^{k+1} \frac{1}{k} \right) \mid n = 1, 2, 3, \dots \right\} &= \\ \left\{ v(1, L_\infty)1, v(2, L_\infty)\left(\frac{1}{2}\right), v(3, L_\infty)\left(\frac{5}{6}\right), v(4, L_\infty)\left(\frac{2}{3}\right), \dots \right\}. \end{aligned}$$

Its kernel is the limit of the corresponding sequence.

$$\begin{aligned} \text{B. } \sum_{n=1}^\infty \frac{1}{n^2} &= \left\{ v(n, L_\infty) \left(\sum_{k=1}^n \frac{1}{k^2} \right) \mid n = 1, 2, 3, \dots \right\} = \\ \left\{ v(1, L_\infty)1, v(2, L_\infty)\left(\frac{5}{4}\right), v(3, L_\infty)\left(\frac{49}{36}\right), v(4, L_\infty)\left(\frac{405}{238}\right), \dots \right\}. \end{aligned}$$

Again, its kernel is the limit of the corresponding sequence.

However, Definition 8 also gives the values of the sums of arbitrary series. This does not eliminate the significance of convergence issues because they are crucial in determining which sums are convertible to real numbers and which ones are to be left as multivalued numbers. The following example gives

two well-known divergent series. For an interesting number theoretic and probabilistic distribution approach to sums of series, see [16].

Example 6. Divergent series:

$$A. \sum_{n=1}^{\infty} \frac{1}{n} = \left\{ v_{(n, L_{\infty})} \left(\sum_{k=1}^n \frac{1}{k} \right) \middle| n = 1, 2, 3, \dots \right\} = \left\{ v_{(1, L_{\infty})} 1, v_{(2, L_{\infty})} \left(\frac{3}{2} \right), v_{(3, L_{\infty})} \left(\frac{11}{6} \right), v_{(4, L_{\infty})} \left(\frac{50}{24} \right), \dots \right\}.$$

The corresponding sequence of this set is divergent.

$$B. \sum_{n=1}^{\infty} (-1)^n = \left\{ v_{(n, L_{\infty})} \left(\sum_{k=1}^n (-1)^k \right) \middle| n = 1, 2, 3, \dots \right\} = \left\{ v_{(1, L_{\infty})} (-1), v_{(2, L_{\infty})} 0, v_{(3, L_{\infty})} (-1), v_{(4, L_{\infty})} 0, \dots \right\} \rightarrow \{-1, 0\}.$$

Again, the corresponding sequence of this multivalued set is divergent.

For other approaches to the sums of divergent series, see [17]. Before closing this section, for the alternative approach employed in this paper, see [18]. There is an interesting account of specialized set theory for analysis [19].

4. Conclusions

Using multivalued logic, one may record the “history” of a set by encoding it in the form of truth values for its elementhood. In the case of constructions of arbitrary sets from finite sets, the elementhood values are to be forgotten to classify them into “false” and “non-false” since the aim is only to indicate the construction process. However, in calculus, elementhood values are essential since they reveal what objects we are ending with.

Besides the usage in sets, this logic provides a way for statements that are not downright wrong to have a chance to be expressed in the form of its truth value, which is neither false nor true, most notably, when it deals with paradoxes such as the Russell paradox. Consequently, forbidden sets that are banned in the ZF set theory reemerge, although not in the form of ordinary sets but as multivalued sets. As a consequence, those sets can no longer be unique because multivalued sets are not determined by their elements alone but by the values of the respective elements.

The adventure with constructing sets from sequences of finite sets opens a new practical and encompassing method of giving truth values, namely by sets and their negatives, more precisely, by their equipollence classes.

Classical results come back in a more “tolerant” nature as they do not necessarily judge things as right and wrong. They reemerge as “kernels,” namely the direction of statements toward the truest values. In addition, more subtle facts and tendencies behind the not-so-true world are revealed.

The main contribution of this paper is to view set theory and calculus using multivalued logic, which

may be rewarding. In addition to the construction of sets from finite ones, it also reveals a general usage of equipollence classes and their negatives as truth values when the elements involved may be too numerous than the real numbers in the interval $[0, 1]$. One of the novel aspects of this paper is the use of multivalued logic in recording abstract constructions of new mathematical objects, a feature that might be worthwhile to be studied algebraically.

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