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Characterization of Non-Orientable Surfaces Using GeoGebra

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Abstract: In this article, a study of the fundamental characteristics of non-orientable surfaces is conducted using GeoGebra. This software is an excellent means to experiment, explore, discover, visualize, and manipulate mathematical objects in a teaching-learning process. This article presents and describes concepts and characteristics of the Möbius band and Klein bottle, which are animated by GeoGebra through the use of sliders, which allow the learning environment to be dynamized to visualize their properties and characteristics. The results of this research provide fundamental tools for non-orientable surfaces through the use of the GeoGebra dynamizing environment, which allows multiple representations of the presented subject matter.

Keywords: non-orientable surfaces, the Klein bottle, the Möbius band, mathematics, the GeoGebra software.

使用地理几何表征不可定向表面

摘要：在本文中，使用地理几何对不可定向表面的基本特征进行了研究。该软件是在教学过程中实验、探索、发现、可视化和操作数学对象的绝佳手段。本文介绍并描述了莫比乌斯带和克莱因瓶的概念和特征，地理几何通过使用滑块对它们进行动画处理，从而使学习环境动态化，以可视化它们的属性和特征。这项研究的结果通过使用地理几何动态环境为不可定向表面提供了基本工具，该环境允许对所呈现的主题进行多种表示。

关键词：不可定向表面、克莱因瓶、莫比乌斯带、数学、地理几何软件。

Introduction

In the Möbius band and the Klein bottle, one can see fundamental properties of topology, which is a fundamental line of research in mathematics found in almost all areas of mathematics, such as algebra, geometry, analysis, and differential equations. Its methods and results facilitate the treatment of various problems applied to the aforementioned areas. Its origin can be placed in the 18th century, because until

that time mathematical problems were linked to the idea of measurement, magnitude, or distance, and at that time, problems began to be posed in which these aspects were no longer important; these problems do not depend on the distance or size but on the place [1], [2], [3], [4].

The Klein bottle is a non-orientable surface that has significant importance in topology and geometry. It was discovered by the German mathematician Felix

Klein in 1882, and the Möbius strip is another non-orientable surface that has intriguing properties and great importance in topology and geometry. It was discovered by the German mathematician August Ferdinand Möbius in 1858.

It is necessary to make an adequate treatment of the topological concepts present in the Möbius band and the Klein bottle in such a way that it can be accessible to the mathematical academic community. Therefore, through the use of GeoGebra, we want to present the processes that are enlivened in the interpretation of concepts, properties, and characteristics of the topological elements in the Möbius band and the Klein bottle through the visualization and representation of animated constructions that allow to rescue the new possibilities of treatment of the mathematical concept that generates clearer and more precise processes.

In the development of this article, the use of the GeoGebra software is promoted, which enables a better use of creativity, sensitivity, experience, maturity, and mathematical knowledge, facilitating the construction of interactive material to induce discovery and help to visualize the results of the analysis and deepen the concepts. The use of the GeoGebra software provides ample possibilities for visualizing, exploring, analyzing, and conjecturing results. The characteristics and properties of the software allow the development of dynamic and interactive geometric constructions, which strengthen the teaching and learning of mathematical conceptualizations [5], [6].

This study of the topological elements in the Möbius band and the Klein bottle through GeoGebra is presented in five sections. In the first section, the importance of the GeoGebra software as a dynamic medium is presented; in the second section, the development of the Möbius band and the Klein bottle is presented; in the third section, the main characteristics of the Klein bottle are presented; in the fourth section, the main characteristics of the Möbius band are presented; in the last section, the simulations and animations in GeoGebra of the Möbius band and the Klein bottle are presented.

1. GeoGebra as a Dynamic Tool

The use of ICT in educational mathematics as a tool to facilitate pedagogical tasks fosters creative capacity and innovation and accelerates the process toward change. This leads to a transformation in teaching environments that favor didactics and playfulness for the motivation and acquisition of different knowledge. The educational use of ICT encourages the development of favorable attitudes toward learning science and technology by using interactive programs and the search for scientific information.

The implementation of ICT in mathematics is an aid in pedagogical training, i.e., they serve as a complement or facilitator in education, and the

resources offered in the preparation of educational material should be used to enhance the cognitive abilities of each individual. ICT in the area of mathematics allows visualization, understood as the ability to represent, transform, generate, communicate, document, and reflect on the visual information generated through the use of technology, the latter being essential for today's life [7].

The mathematical assistant GeoGebra integrates the work in the areas of geometry, algebra, and mathematical analysis in a dynamic environment enhancing the development of variational thinking. In this sense, by recreating dynamic environments, the software allows users to visualize and represent variation relationships using sliders. Based on the above, this software can be assumed to be a didactic tool since it is a physical or symbolic element that, within a learning environment, provides tools for the presentation of a particular subject, and at the same time provides the user with a form of representation, visualization, and organization of the concepts worked on in the study of certain mathematical objects [8].

The use of this dynamic geometry program allows the approach to geometry and other aspects of mathematics through experimentation and manipulation of different elements, facilitating the realization of constructions to deduce results and properties from direct observation. The use and applicability of the software has been the focus of several researchers in the field. Godoy states that "GeoGebra is an educational software that allows experimental learning and discovery, where the designer creates rich environments in situations that the user can explore, i.e., they can build their elements and draw conclusions according to certain properties" [9]. The student must arrive at knowledge from experiences by creating their own models of thought and interpretations of the problem, so that it provides an adequate means for our goal.

According to Espina, "the software allows to perform dynamic constructions in an easy and intuitive way" [10]. In this sense, students can work with this application in an interactive and simple way, affirming that it is not a complicated process and that, in addition, extensive sections are not required for its explanation and operation. These observations guide us to the use of dynamic geometry software as a means of visualization to verify concepts, characteristics, and properties of these mathematical objects through dynamic constructions.

The virtues of GeoGebra are strengthened by the visualization processes that it provides through its dynamic character, which is why we present the views of researchers on these processes. Arcavi defines visualization as "the ability, process, and product of the creation, interpretation, use, and reflection on figures, images, and diagrams in our mind, on paper, or with

technological tools for representing and communicating information, thinking, developing ideas, and advancing understanding” [8].

The main characteristic attributed by Arcavi to visualization is that it offers a method of seeing the invisible; hence, many people believe that visualization is an innate ability and a matter that should remain on the margins of educational activity [9]. However, in our case, it plays a fundamental role in understanding the concepts, characteristics, and properties of the Möbius strip and the Klein bottle, given the processes of manipulating, experimenting, and generating visual conjectures through the use of the GeoGebra software.

The visualization made possible with dynamic geometry software allows the user not only to see but also to explore mathematical and conceptual relationships that can be difficult to “understand” when not using technological resources. This is a main reason why the incorporation of resources such as GeoGebra in a teaching environment is necessary [11-14].

In our proposal, visualization would be associated with the topological elements in the Möbius strip and the Klein bottle, which are presented for the understanding of concepts, properties, and characteristics. In this sense, with the application of the GeoGebra software, we intend that, from the elements designed in this tool, visualization processes are achieved for the understanding and construction of knowledge to address the topic presented.

2. Klein’s Bottle and the Möbius Strip

The Klein bottle is a non-orientable surface that has significant importance in topology and geometry. It was discovered by the German mathematician Felix Klein in 1882 [15].

The Klein bottle can be constructed by the interesting property of identifying points in a plane. Imagine taking a rectangle and gluing the opposite edges in a special way: by gluing an edge inverted and the other not inverted. Then, if one curves the rectangle into a cylinder shape and connects the ends of this cylinder in a twisted shape, they obtain the Klein bottle.

One of the most fascinating properties of the Klein bottle is that it is a single-sided surface without edges. This means that if one was to move along the surface, they would eventually return to the starting point, but with the orientation reversed. It is as if one could walk across the Klein bottle without encountering a division between the inside and outside.

The importance of the Klein bottle lies in its role in topology theory and the understanding of non-orientable varieties. It helps mathematicians understand how surfaces can be constructed and classified and explore the properties of more complex topological spaces. In addition, the Klein bottle has inspired research in other fields, such as physics and art,

because of its uniqueness and intriguing properties.

In short, the Klein bottle is a non-orientable surface that defies spatial intuition. Its study and understanding have contributed to the development of topology and geometry, and it has captured the imagination of many people because of its peculiar structure and its importance in various fields of knowledge [16-19].

On the other hand, the Möbius band is another non-orientable surface that has intriguing properties and great importance in topology and geometry. It was discovered by the German mathematician August Ferdinand Möbius in 1858 [20], [21].

The Möbius strip can be constructed by taking a strip of paper and joining its ends after giving one of them a 180-degree turn before joining them together. This results in a surface with a single face and edge. Surprisingly, by traversing the surface of the Möbius strip, one will arrive back at the starting point, but with the orientation reversed. That is, if one was to draw a line along the band, they would eventually return to the original line but would have made a complete turn.

The importance of the Möbius strip lies in its role in topology and in understanding non-orientable surfaces. It serves as a concrete and visually accessible example of a surface with unique characteristics and has been central to the development of topology theory. In addition, the Möbius strip has found applications in various fields, such as physics, chemistry, and art, because of its special symmetry and interesting mathematical properties.

In summary, the Möbius strip is a non-orientable surface that defies our geometric intuition. Its study has contributed to the understanding of topological varieties, and its uniqueness and properties have inspired research and applications in various fields. This is an emblematic example of how abstract mathematical concepts can be manifested in tangible and visual forms.

3. Main Characteristics of the Klein Bottle

Below, we present the fundamental characteristics of the Klein bottle visualized through GeoGebra. These characteristics are visualized in Fig. 2, 3, 8-13.

3.1. Non-Orientability

This characteristic can be demonstrated using the Gauss-Bonnet theorem. In particular, the Gauss-Bonnet theorem states that the total curvature of a closed surface is related to its topology. In the case of the Klein bottle, the total curvature is zero, implying that the surface is not orientable.

3.2. Self-Intersection Point

The existence of a self-intersection point can be demonstrated using knot theory. In particular, it can be shown that the Klein bottle is homeomorphic to a knot

known as the Klein knot. This knot has a self-intersection point, which can be visualized as a point where the surface appears to be cut into itself.

3.3. Quartic Symmetry

The quartic symmetry of the Klein bottle can be demonstrated using the parametric equations of the surface. In particular, if a 90-degree rotation is applied to the parametric equations, the surface looks the same after rotation.

3.4. Two Inner and Two Outer Sides

The existence of two inner and two outer sides can be shown using the surface normal property. In particular, the surface normal changes direction twice as it traverses a closed curve around the Klein bottle, implying that there are two inner sides and two outer sides.

3.5. Surface with an Edge

The Klein bottle is an edged surface because it has an edge that can be traced. This can be shown using the topological definition of an edge surface, which states that an edge surface is homeomorphic to a disk with a finite number of points removed. The Klein bottle can be constructed from a disk with a finite number of points removed and has an edge that can be traced.

4. Main Characteristics of the Möbius Band

Below, we present the fundamental characteristics of the Möbius band visualized through GeoGebra. These characteristics are shown in Fig. 1, 2, 4-7.

4.1. Non-Orientability

This feature can be demonstrated using arguments based on the cover theory. One can consider a universal cover, such as the Euclidean plane, and show that there is no projective continuous application covering the Möbius band without generating contradictions. The above implies that one cannot assign a consistent orientation to the Möbius band.

4.2. Self-Intersection

One can prove the self-intersection of the Möbius band by using a suitable parameterization and studying the intersection of curves or surfaces. A viable way to do this is to perform a parametrization of the Möbius band and analyze the points where the coordinates cancel or where a crossing occurs in that parametrization. This proves that the Möbius band intersects with itself at a point.

4.3. Symmetry

The symmetry of the Möbius band can be proved by starting from the fact that for any point on the band, it is possible to find another point symmetric with respect

to an axis of symmetry. This can be done using geometrical or algebraic arguments. As an example, one can consider a reflection with respect to a plane or a linear transformation that preserves the properties of the Möbius band.

5. Importance of Using GeoGebra in the Klein Bottle and Möbius Band

GeoGebra is a versatile tool that allows you to explore and interactively visualize many features of the Möbius strip and Klein bottle. Here are some features that can be addressed with GeoGebra that may be more difficult to achieve with other means:

5.1. Interactive Representation

GeoGebra allows you to create three-dimensional models of the Möbius strip and Klein bottle, making it easy to visualize and understand their structure in an interactive three-dimensional environment. Students can rotate, zoom in, and zoom out the models to explore different perspectives and viewing angles to understand the characteristics of these non-orientable surfaces, which are not easy to work with in a different environment, as shown in Fig. 1-3.

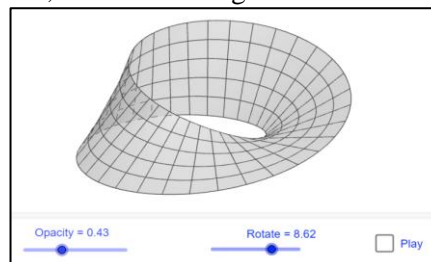


Fig. 1 The Möbius strip in GeoGebra (Made by the authors)

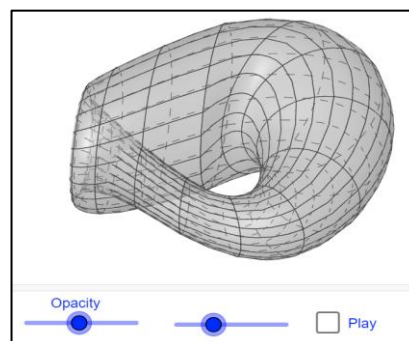


Fig. 2 The Klein bottle in GeoGebra (Made by the authors)

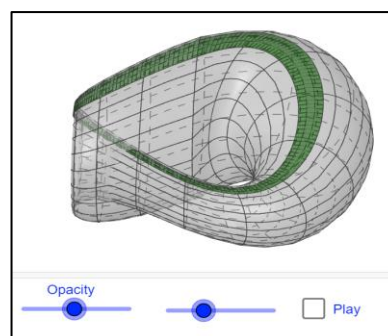


Fig. 3 The Klein bottle and Möbius strip in GeoGebra (Made by the authors)

5.2. Animations and Transformations

Through GeoGebra, it is possible to create animations that illustrate the transformations and dynamic properties of the Möbius strip and Klein bottle. For example, transformations can be shown when cutting the Möbius strip or when turning on the Klein bottle. These animations help students to visualize and better understand the topological changes and perform demonstrations of such features using theoretical tools, as shown in Fig. 4-13.

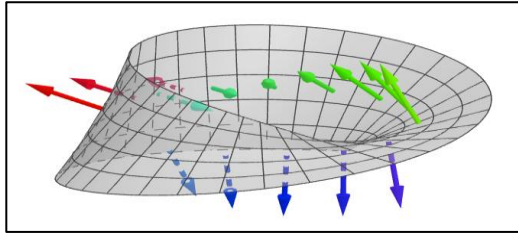


Fig. 4 Animation 1 of the Möbius strip in GeoGebra (Made by the authors)

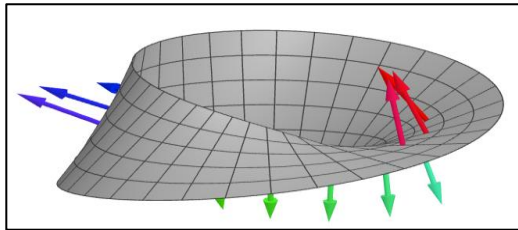


Fig. 5 Animation 2 of the Möbius strip in GeoGebra (Made by the authors)

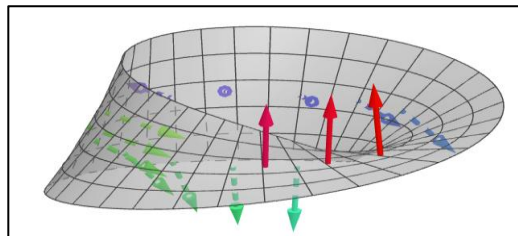


Fig. 6 Animation 3 of the Möbius strip in GeoGebra (Made by the authors)

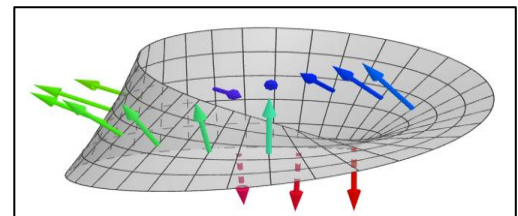


Fig. 7 Animation 4 of the Möbius strip in GeoGebra (Made by the authors)

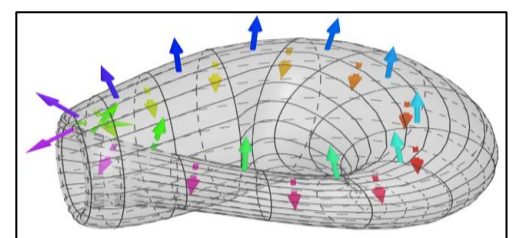


Fig. 8 Animation 1 of the Klein bottle in GeoGebra (Made by the authors)

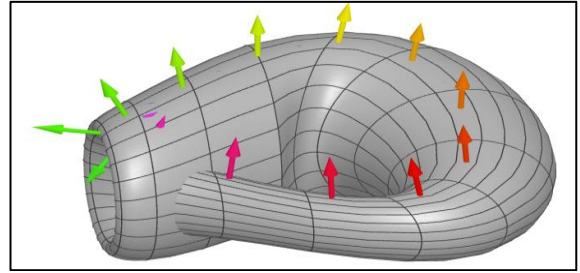


Fig. 9 Animation 2 of the Klein bottle in GeoGebra (Made by the authors)

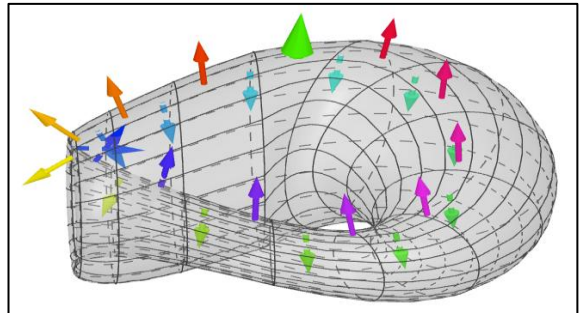


Fig. 10 Animation 3 of the Klein bottle in GeoGebra (Made by the authors)

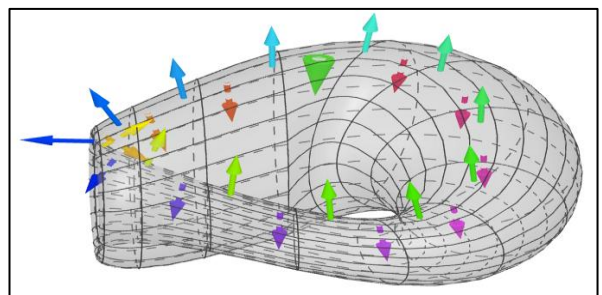


Fig. 11 Animation 4 of the Klein bottle in GeoGebra (Made by the authors)

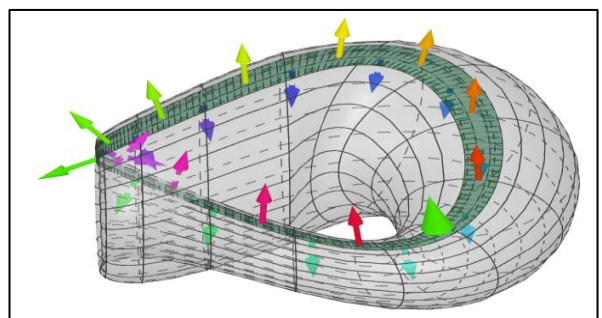


Fig. 12 Animation 1 of the Klein bottle and Möbius strip in GeoGebra (Made by the authors)

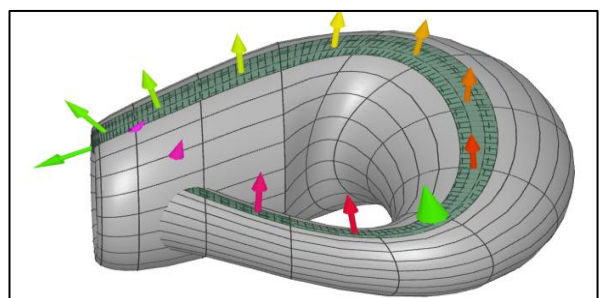


Fig. 13 Animation 2 of the Klein bottle and Möbius strip in GeoGebra (Made by the authors)

5.3. Parameterization and Equations

GeoGebra allows you to define equations and parameters to represent Möbius strip and Klein bottle surfaces. This makes it easy to manipulate the parameters and observe how they affect the shape and properties of the surfaces. Students can experiment with different equations and see the effects in real time, which is a critical element in understanding concepts that are difficult to visualize without the aid of a tool such as GeoGebra, as shown in Fig. 14-16.

$$\text{Möbius} = \begin{pmatrix} 1.8 \left(1 + \frac{v}{2} \cos\left(\frac{u}{2}\right) \right) \cos(u) \\ 1.8 \left(1 + \frac{v}{2} \cos\left(\frac{u}{2}\right) \right) \text{sen}(u) \\ 1.8 \cdot \frac{v}{2} \text{sen}\left(\frac{u}{2}\right) \end{pmatrix}$$

Fig. 14 Parameterization and the Möbius strip equations (Made by the authors)

$$\text{Klein0} = \begin{pmatrix} 0.9 \left(3(1 + \text{sen}(u)) + 2 \left(1 - \frac{\cos(u)}{2} \right) \cos(v) \right) \cos(u) \\ 0.9 \left(4 + 2 \left(1 - \frac{\cos(u)}{2} \right) \cos(v) \right) \text{sen}(u) \\ 0.9 \cdot 2 \left(1 - \frac{\cos(u)}{2} \right) \text{sen}(v) \end{pmatrix}$$

Fig. 15 Parameterization and the Klein bottle equations 1 (Made by the authors)

$$\text{Klein1} = \begin{pmatrix} 0.9 \left(3(1 + \text{sen}(u)) \cos(u) - 2 \left(1 - \frac{\cos(u)}{2} \right) \cos(v) \right) \\ 0.9 \cdot 4 \text{sen}(u) \\ 0.9 \cdot 2 \left(1 - \frac{\cos(u)}{2} \right) \text{sen}(v) \end{pmatrix}$$

Fig. 16 Parameterization and the Klein bottle equations 2 (Made by the authors)

5.4. Related Surfaces

GeoGebra also allows one to explore and relate the Möbius strip and Klein bottle to other mathematical surfaces, such as the projective plane and/or the torus. Students can build visual connections and understand the relationships between these surfaces through interactive representations, as shown in Fig. 17-20.

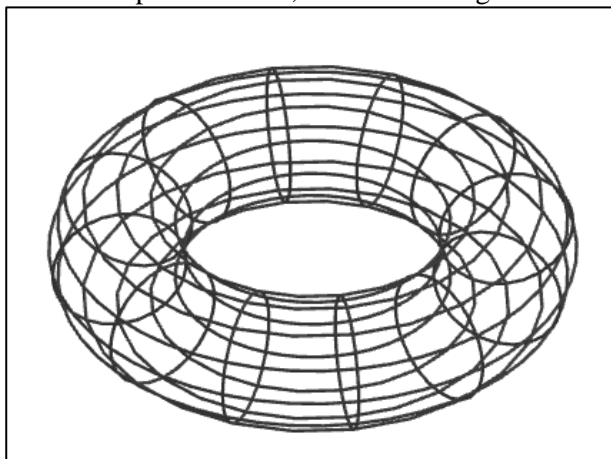


Fig. 17 Torus in GeoGebra (Made by the authors)

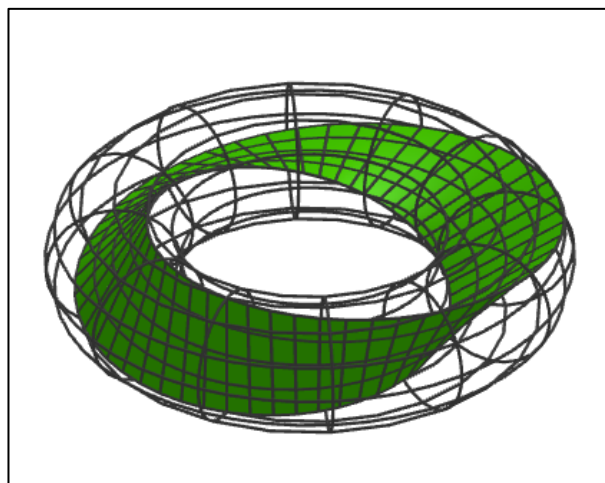


Fig. 18 Torus and the Möbius strip in GeoGebra (Made by the authors)

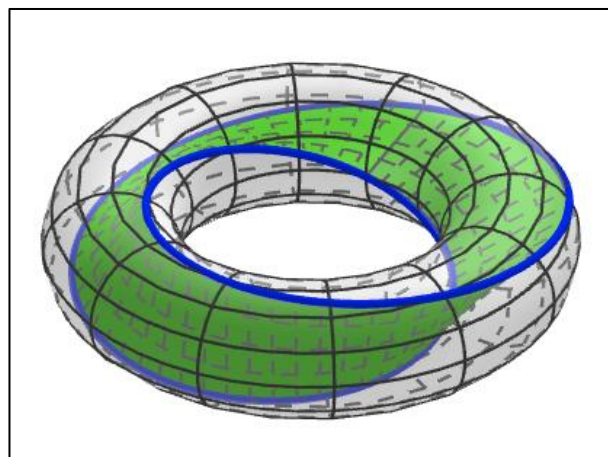


Fig. 19 Torus and the Möbius strip in GeoGebra 2 (Made by the authors)

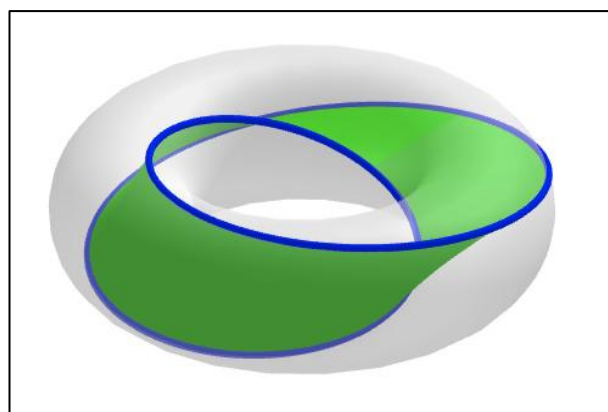


Fig. 20 Torus and the Möbius strip in GeoGebra 3 (Made by the authors)

6. Conclusion

Based on the interaction with the GeoGebra software through the construction of its applets for the visualization of concepts, properties, and characteristics of non-orientable surfaces, it is initially concluded that the geometric character allowed the construction of dynamic applets in GeoGebra for the visualization of the fundamental elements of the

Möbius strip and Klein bottle.

The constructions in GeoGebra are a means that make possible the understanding of some concepts, properties, and characteristics immersed in these surfaces since they propitiate the processes of visualization, experimentation, generation, and validation of visual conjectures that contribute to the production of knowledge.

These animations correspond to an application created in GeoGebra that allows the interactive and visual illustration of the concepts, characteristics, and properties of the most important non-orientable surfaces of topology. By their nature, these applications are appropriate for activities of intuitive exploration of mathematical concepts and ideas that, in a certain sense, become complex.

Through this research, the use of GeoGebra is strengthened as a dynamic medium in the teaching and learning process of topological concepts of non-orientable surfaces. With the above, we emphasize that this software is fundamental in understanding the characteristics of non-orientable surfaces, which can be observed through the fundamental tools offered by GeoGebra.

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