湖南大学学报（自然科学版）
Journal of Hunan University (Natural Sciences)

第 50 卷 第 10 期
2 0 2 3 年 10 月

Available online at http://jonuns.com/index.php/journal/index

Vol. 50 No. 10
October 2023

Open Access Article https://doi.org/10.55463/issn.1674-2974.50.10.19

Some Results in Calculus and Set Theory through the Directed Set Method

Ravi Ahmad Salim*, Ernastuti

Department of Information System, Faculty of Computer Science, Gunadarma University, Jl. Margonda Raya 100, Depok, 16424, Indonesia

* Corresponding author: ravi@staff.gunadarma.ac.id

Received: July 25, 2023 / Revised: August 13, 2023 / Accepted: September 18, 2023 / Published: October 31, 2023

Abstract: So far, real numbers have not dealt with sums of arbitrary series or given answers to every infinite operation application like 5 5 5 5 5 5 5 5 5. They surrender to the paradigm that not every continuous function is differentiable, and no one is Riemann integrable. Likewise, set theory does not hint at existing number-like entities that are as many as ordinals but retain field axioms or a set universe expressed as functions on natural numbers with finite sets as its values, which is very unfortunate as there is no analysis for such entities. This paper aims to unravel the blockades that hinder the solutions to such problems through something described as follows. Suppose in a world of sequences of balls, every single color from red, green, and blue appears separately. They are evaluated term by term by how they end up, and such a sequence is "green" if, in the end, all of them are green balls. Such a sequence is darker than another if the former is eventually always darker than the latter. However, if red and green boxes appear one after another forever, then it is more appropriate to describe them as red–green, namely "yellow," a color previously unknown. The above is how new colors occur in the presence of endless sequences of colored balls. Such new objects would serve as solutions to the above problems. This research aims to supply calculus with more than just real numbers, called constructed numbers, so that undefined notions such as undefined derivatives or integrals, divergent sequences, can be given values in the hope that through these values, they can be further studied in more detail. Another goal is to explore the Zemelo–Fraenkel set theory to view it one step closer to the authors' conjecture that everything in its universe and mathematics are expressible as sequences of finite structures. Sets defined there are likewise called constructed sets. Objects such as constructed numbers or sets they called directed set objects or constructed objects. These objects may not be completely tangible, but like a telescope that can zoom the moon's surface as sharply as anybody wishes but will never touch it. The directed set method that produces constructed objects is the novelty of this study.

Keywords: directed set objects, constructed derivatives, constructed Riemann integrals, sums of arbitrary series.

透過有向集合方法得出的微積分和集合論的一些結果

摘要：到目前為止，實數還沒有處理任意級數的和，也沒有給出像5×2-3÷4+⋯這樣的無限運算應用的答案。他們屈服於這樣一種範式，即並非每個連續函數都是可微的，並且沒有一個函數是黎曼可積的。同樣，集合論並不暗示現有的類別數實體與序數一樣多，但保留了場公理或集合宇宙，表示為以有限集合為值的自然數上的函數，這是非常不幸的，因為沒有分析此類實體。本文旨在透過以下描述的內容來解開阻礙解決此類問題的障

礙。假設在一個球序列的世界中，紅色、綠色和藍色中的每種顏色單獨出現。它們根據它們的最終結果逐項評估，如果最終它們都是綠球，那麼這樣的序列就是「綠球」。如果前者最終總是比後者更暗，則這樣的序列比另一個序列更暗。然而，如果紅色和綠色的盒子永遠相繼出現，那麼將它們描述為紅綠，即"黃色"，一種以前未知的顏色更合適。以上是在無窮無盡的彩球序列存在的情況下新顏色是如何出現的。這些新物體將作為上述問題的解決方案。這項研究的目的是為微積分提供不僅僅是實數（稱為構造數），以便可以為未定義的概念（例如未定義的導數或積分、發散序列）賦予值，希望透過這些值可以對它們進行更深入的研究。細節。另一個目標是探索澤梅洛弗蘭克爾集合論，使其更接近作者的猜想，即宇宙和數學中的一切都可以表達為有限結構的序列。在那裡定義的集合同樣被稱為構造集合。諸如構造數或集合之類的物件稱為有向集合物件或建構物件。這些物體可能不是完全有形的，但就像一架望遠鏡，可以隨心所欲地放大月球表面，但永遠不會觸摸它。產生構造物件的有向集合方法是本研究的新穎之處。

关键词：有向集合物件、建構導數、建構黎曼積分、任意級數之和。

1. Introduction

There are no real numbers to answer questions: What is the sum of the series $\sum_{n=1}^{\infty} n^2$? What is the answer to infinitely many operations like $a_1 \times a_2 + a_3 - a_4 \cdots$ from left to right? What is the derivative of the characteristic function $\chi_Q(x)$ of rational numbers? What is the value of $\int_0^{\infty} \sin\frac{1}{x}dx$? This paper tries to provide answers that resemble real numbers to such questions while exploring things related to ordinal numbers and set theory. The *directed set method* employed here is an elaboration of allowing things to change over time and turns our discussion from ordinary objects to varieties of possible change tracks with characterizations for each change track according to its behavior near the end. Several issues must be settled beforehand. First, what constitutes "time" should be decided that is similar to the indices of a sequence. To talk abstractly, this could be a total ordering or even a lattice without a maximal element. As it will be clear after a while, the existence of a maximal element inhibits the emergence of novel elements or objects, and the introduction of time would be worthless. However, for the current paper, positive real numbers R^+ or positive integers N suffice to be used as "time" since what is needed is something endless, like the progression of the integers approaching infinity. However, real numbers are preferable to integers when dealing with analysis. Also, the former includes the latter. Hence, we decide R^+ the positive reals for presentation.

Second, time, now R^+, forms paths in a mathematical structure, for example, real numbers or sets. If new elements are desired, then characterization criteria are needed to determine the properties and relations of the new objects, including the reason why they are unequal to old ones. The criteria are represented by time-dependent sets S_t of objects where $t > 0$ so that a function x_t with range $G \overset{\text{def}}{=} \bigcup_{t>0} S_t$ can be evaluated whether $x_t \in S_t$ or $x_t \notin S_t$ at each time t. For example, a predicate P with variable x, namely, a sentence about an indeterminate x, is defined *true* for the function x_t if the statement "$P(x_t)$" is *eventually true*; namely, there is $t_0 > 0$ so that $\{x_t | P(x_t)\} \subseteq S_t$ for $t > t_0$. A similar situation holds for predicates with more than one indeterminate when the latter are replaced by functions with range G: "$P(x_t, y_t)$" is *eventually true* if there is t_0 so that $\{(x_t, y_t) | P(x_t, y_t)\} \subseteq S_t \times S_t$ for $t \geq t_0$. If a property or object is not eventually true, then it is an altogether new property or object, or it may be *recurring*, as discussed in the next section.

If the "time" has a maximal element, say 0.9, then the concept of eventual truth means "true at 9," which allows no space for new properties or objects to emerge. For example, let $x_t = \frac{1}{t}$ and $S_t = (t, \infty)$, for $t > 0$. Then $P(x) = "0 < x < r"$ is true for any $r > 0$ since for $t > r$, $0 < x_t < r$. Thus, x is a novel object with previously unknown characteristics. If $S_t = (0,9]$ then $x = \frac{1}{9}$ is true since $x_9 = \frac{1}{9}$ is true for $t \geq 9$. But we already have $\frac{1}{9}$ before, so there is nothing new.

For another example, let $f: R^+ \to U$ be a function with $U = \{\{0,1,2,\cdots,n\} | n = 1,2,\cdots\}$, $S_t = f\big((t,\infty)\big)$ for $f(s) = \{0,1,2,\cdots,\lceil s \rceil\}$ where $\lceil s \rceil$ denotes the smallest integer p so that $p \geq s$. Then, the sentence "$(\forall n \in N)(n \in f)$," which states that "every natural number is a member of f" holds since for each integer $n \geq 0$, $n \in f(t)$ while $f(t) \in S_t$ for any $t \geq n$. In addition, the sentence$P(f)=$"$g \in f$" holds for $g(t)=\ln\lceil t\rceil$, because $g(t) \in f(t) \in S_t$ for any $t>0$. Thus, f can be the set of all natural numbers with additional elements, *not* integers, such as g above, because $\ln n$ is never an integer. Objects such as the additional elements of f are new beings because they are not equal to anything known before, hence the name "non-standard elements" for a nice obvious approach to non-standard analysis [1] and a category-theoretic approach to non-standard analysis [2].

By introducing criterion sets using R^+, we present new beings with new characteristics. In the following sections, we introduce a new construct of derivatives in calculus. The derivative is defined for arbitrary actual functions, although they are not always real. Integrals have similar traits. In set theory functions on R^+ with finite sets as values can handle infinite sets as non-standard objects, their power sets, and countably infinitely repeated applications of power set operation. It expects how gigantic a set can be from sequences of finite sets through this method.

2. Basic Theory

This section is devoted to formalizing the discussion of the last section a step further and to lay some foundations and present some previous results for further discussions.

Definition 1: Directed set, parameters, truth directing.

A. Let G be a non-empty set. A directed set P over G is a tuple (G,D) where $D=\{S_t\}_{t>0}$ is a family of sets $S_t \subseteq G$ and $G=\bigcup_{t>0} S_t$. D is called the direction of P, whereas G is called its ground (set). In this situation, D is called the direction for G.

B. The collection of all variables that take values in G is called the parameter of P. However, oftentimes, it is more convenient to view the parameter as a variable of G as it is aimed only to standardize the variables on G.

C. Let $Q(x_1,x_2,\cdots,x_n)$ be a predicate with indeterminates $x_1,x_2,\cdots,x_n$. Let $f_i: R^+ \to G, i=1,2,\cdots,n$. $Q(f_1,f_2,\cdots,f_n)$ is defined to be (eventually) true if and only if there is a number $t_0>0$ so that $f_1(t),f_2(t),\cdots,f_n(t) \in S_t$ and $Q\big(f_1(t),f_2(t),\cdots,f_n(t)\big)$ is satisfied for all $t \geq t_0$.

D. Let $Q(x_1,x_2,\cdots,x_n)$ be a predicate with indeterminates $x_1,x_2,\cdots,x_n$. Let $f_i: R^+ \to G, i=1,2,\cdots,n$. $Q(f_1,f_2,\cdots,f_n)$ is defined to be recurring if and only if for any natural number K there is a number $t_0>K$ so that $f_1(t_0),f_2(t_0),\cdots,f_n(t_0) \in S_{t_0}$ and $Q\big(f_1(t_0),f_2(t_0),\cdots,f_n(t_0)\big)$ is satisfied.

E. Objects obtained through the above procedures are called directed set objects or simply constructed objects.

For example, if p is a parameter of the directed set $(R^+,\{(t,\infty)\}_{t>0})$, the predicate "$\sin p \in [-1,1]$" is eventually true. The predicate "$\sin p = q$" is recurringly true for $q \in [-1,1]$ false for other values of q.

A common variable, such as a real number variable x can be viewed as a parameter of the directed set $(R,\{S_t\}_{t>0})$ where $S_t=R$ for all $t>0$. Meanwhile, a constant c can be viewed as the parameter for the directed set $(\{c\},\{S_t\}_{t>0})$, where $S_t=\{c\}$.

The following is a general statement about the effect of a function on directed sets. The proofs are omitted because of their straightforward nature.

Theorem 2: Induced map on direct sets restriction or expansion of directions.

A. Let D be a direction for G and $f: G \to H$ be a function where H is a non-empty set. Then $f(D) \stackrel{\text{def}}{=} \{f(S_t)\}_{t>0}$ is a direction for $f(G)$. The induced map preserves the partial ordering of the direction D.

B. Let $D=\{S_t\}_{t>0}$ be a direction for G, then $D_t \stackrel{\text{def}}{=} \{S_u\}_{u>t}$ is a direction for $G_t \stackrel{\text{def}}{=} \bigcup_{u>t} S_t$.

Part B of Theorem 2 states that the ground and its direction are relative to contexts and can be adjusted arbitrarily if the "tail" remains the same. Hence a parameter does not fully depend on the ground of its directed set because depends only on the tail of its direction.

Theorem 13 in [3] uses the symbol $R_{-\infty}$ for the parameter of the directed set $\big((-\infty,0),\{(-\infty,-t)\}_{t>0}\big)$, R_a for $\left((a,\infty),\left\{\left(a,a+\frac{1}{t}\right)\right\}_{t>0}\right)$, L_a for $\left((-\infty,a),\left\{\left(a,a+\frac{1}{t}\right)\right\}_{t>0}\right)$, and L_∞ for $\big((0,\infty),\{(t,\infty)\}_{t>0}\big)$. We cite the theorem as follows:

Theorem 3: Interrelations between parameters. If a and b are real numbers, unless specified, then:

a. $-L_a=R_{-a}$ and $-R_a=L_{-a}$. Also: $-L_\infty=R_{-\infty}$ and $-R_{-\infty}=L_\infty$.

b. $\frac{1}{R_0}=L_\infty$ and $\frac{1}{L_0}=R_{-\infty}$. Also, $\frac{1}{R_{-\infty}}=L_0$ and $\frac{1}{L_\infty}=R_0$.

c. If $b \neq 0$, $\frac{1}{L_b}=R_{\frac{1}{b}}$ and $\frac{1}{R_b}=L_{\frac{1}{b}}$.

d. $a+R_b=R_a+R_b=R_{a+b}$ and $a+L_b=L_a+L_b=L_{a+b}$.

e. $a-L_b=R_a-L_b=R_{a-b}$ and $a-R_b=L_a-R_b=L_{a-b}$.

f. If $a,b>0$ then $aR_b=R_aR_b=R_{ab}$ and $aL_b=L_aL_b=L_{ab}$.

g. If $a,b<0$ then $aR_b=R_aR_b=L_{ab}$ and $aL_b=L_aL_b=R_{ab}$.

h. If $a<0$ but $b>0$ then $aR_b=L_aR_b=L_{ab}$ and $aL_b=R_aL_b=R_{ab}$.

i. If $a>0$ but $b<0$ then $aR_b=R_{ab}$ and $L_aR_b=L_{-ab}$. Also $aL_b=L_{ab}$ and $R_aL_b=R_{-ab}$.

j. If $a,b>0$, $\frac{a}{R_b}=L_{\frac{a}{b}}$ and $\frac{R_a}{R_b}=L_{\frac{a}{b}}$. Also, $\frac{a}{L_b}=L_{\frac{a}{b}}$ and $\frac{L_a}{L_b}=R_{\frac{a}{b}}$.

k. If $a, b < 0$, $\frac{a}{R_b} = L_{\frac{a}{b}}$ and $\frac{R_a}{R_b} = L_{\frac{a}{b}}$. Also, $\frac{a}{L_b} = R_{\frac{a}{b}}$ and $\frac{L_a}{L_b} = R_{\frac{a}{b}}$.

l. If $a < 0,\ b > 0$, $\frac{a}{R_b} \le L_{\frac{a}{b}} \le \frac{a}{L_b}$ and $\frac{R_a}{R_b} \le R_{ab} \le \frac{L_a}{R_b}$. Also, $\frac{a}{L_b} \le R_{\frac{a}{b}} \le \frac{a}{R_b}$ and $\frac{L_a}{L_b} \le L_{ab} \le \frac{R_a}{L_b}$.

m. If $a > 0, b < 0$, $L_{\frac{a}{b}} = \frac{a}{R_b}$ and $\frac{R_a}{R_b} \le R_{\frac{a}{b}} \le \frac{R_a}{L_b}$ Also, $R_{\frac{a}{b}} = \frac{a}{L_b}$ and $\frac{L_a}{L_b} \le L_{\frac{a}{b}} \le \frac{L_a}{R_b}$.

Also, in [3], there is Theorem 14 cited below as

Theorem 4: Effect of monotone function on parameters. If f is a function and a is in a neighborhood in the domain of f, then:

a. If f is monotone increasing then $f(L_a) = L_{f(b)}$ and $f(R_a) = R_{f(a)}$. Also $f(L_\infty) = L_{\lim_{x\to\infty} f(x)}$ and $f(R_{-\infty}) = L_{\lim_{x\to-\infty} f(x)}$.

b. If f is monotone decreasing then $f(L_a) = R_{f(b)}$ and $f(R_a) = L_{f(a)}$. Also $f(L_\infty) = R_{\lim_{x\to\infty} f(x)}$ with $\lim_{x\to\infty} |f(x)| < \infty$ and $f(R_{-\infty}) = L_{\lim_{x\to-\infty} f(x)}$ where $\lim_{x\to-\infty} |f(x)| < \infty$.

The following definition defines infinitely many applications of operations that are instrumental to ordinal-like and von Neumann universe-like constructions in subsequent sections.

Definition 2: Infinite actions of operations. Let $\bar{a} = (a_1, a_2, \cdots, a_n, \cdots)$ be a sequence of elements and $\bar{*} = (*_1, *_2, \cdots, *_n, \cdots)$ be a sequence of binary operations. Suppose f is the function

$$f(t) = \prod_{k=1}^{\lceil t\rceil} a_k \stackrel{\text{def}}{=} \left(\left(\cdots\left((a_1 *_1 a_2) *_2 a_2\right) *_3 \cdots\right) *_{\lceil t\rceil} a_{\lceil t\rceil}\right)$$

where $\lceil t\rceil$ is the smallest integer $k \ge t$. Then $\bar{*}(\bar{a})$ is defined to be $f(L_\infty)$. Because it is a step function, the function f is sometimes written by abuse of notation in sequential notation $f = (f(1), f(2), f(3), \cdots)$.

In the case where there is an occurrence of a unary operation, say $F(x)$, then one can define $F^*(x, x) \stackrel{\text{def}}{=} F(x)$. In case there is an occurrence of a ternary operation, say $G(x, y, z)$, then one can split G into $G_2\big(G_1(x, y)\big), z) \stackrel{\text{def}}{=} G(x, y, z)$. Therefore, Definition 2 is in fact much more general than it seems.

3. Research Method

This section deals with seemingly disparate topics, but it will be seen later as of one spirit. Some notations on variables are required.

3.1. Additional Results and Definitions

The definition below handles the symbolisms and terminologies of variables.

Definition 3: Variable over a set, pre-parameters, set operations on variables.

A. Let u be a variable that takes values from the set U, and none otherwise. Then U is said to be the range of u, and u is said to be a variable over U. This is written as $Range(u) = U$. The set of all variables ranging over U is called the pre-parameter over U. Its notation is $var(U)$. U is also called the range of $var(U)$, as often it is more convenient to view it as a variable over U since this is only to standardize variables over U.

B. By abuse of notation, if $S = \{c\}$ then $var(\{c\})$ is written as $var(c)$ or even just c. $var(c)$ is called a singleton variable.

C. Set-theoretic notations are applied to pre-parameters to mean new pre-parameters that range over the corresponding applications to their respective ranges. Therefore, for $var(u)$ and $var(V)$, their union $var(U) \cup var(V)$ is $var(U \cup V)$. Similarly, the meet $var(U) \cap var(V)$ is defined as $var(U \cap V)$, and so on.

Hence, according to Definition 3, $\bigcup_{u\in U} var(u) = var(U)$, since $\bigcup_{u\in U} var(u) = var(\bigcup_{u\in U}\{u\}) = var(U)$.

For parameters of directed sets, the application of set theoretic operation adapts to

Definition 4: Set operations on parameters over directed sets. Let $D = \{S_t\}_{t>0}$ and $E = \{U_t\}_{t>0}$ be directions with grounds S and U respectively. Let s and u be their respective parameters. Then $s \cup t$ has direction $F = \{S_t \cup U_t\}_{t>0}$ and ground $G = S \cup U$. While $s \cap t$ has direction $F = \{S_t \cap U_t\}_{t>0}$ and ground $G = S \cap U$.

The following is to add some more results to Theorems 3 and 4 above.

Theorem 5: Further properties of the parameters. Let a and b be real numbers. Then:

a. $L_a + R_b = var\{a\}$.
b. $L_a R_b = var\{ab\}$.
c. $L_a - R_b = R_a - R_b = var\{a + b\}$.
d. $L_a \cup a \cup R_a = var\{a\}$.
e. $L_a \cap R_b = \begin{cases} var\emptyset, & if\ a \le b, \\ var\{a\}, & if\ a > b, \\ var(b, a), & if\ b < a. \end{cases}$
f. $R_{-\infty} + L_\infty = var(R)$.

Proof:

a. The parameters L_a and R_b have generic elements of their directions $\left(a - \frac{1}{t}, a\right)$ and $\left(b, b + \frac{1}{t}\right)$ respectively. Therefore, their sum at t is $\left(a + b - \frac{1}{t}, a + b + \frac{1}{t}\right)$. Hence, $L_a + R_b$ has a direction that meets in the form of $\{a + b\}$. Hence, the end point $a + b$ determines everything. Therefore, $Range(L_a + R_b) = \{a + b\}$.

b. The parameters L_a and R_b have generic elements of their directions $\left(a - \frac{1}{t}, a\right)$ and $\left(b, b + \frac{1}{t}\right)$, respectively. Therefore, their product at t is (u, v) where $u = \min\left(\left(a - \frac{1}{t}\right)b, a\left(b + \frac{1}{t}\right)\right)$ and $v = \max\left(\left(a - \frac{1}{t}\right)b, a\left(b + \frac{1}{t}\right)\right)$. But $\left(a - \frac{1}{t}\right)b = ab - \frac{b}{t}$

and $a\left(b+\frac{1}{t}\right)=ab+\frac{a}{t}$. Hence, the end point ab determines everything. Therefore, $Range(L_aR_b)=\{ab\}$.

c. By Theorem 2.2 and Part b, $L_a-L_b=L_a+R_b$ which by Part a, has range $\{a+b\}$.

d. At time $t>0$, $Range(L_a\cup a\cup R_a)=\left(a-\frac{1}{t},a\right)\cup\{a\}\cup\left(a,a+\frac{1}{t}\right)=\left(a-\frac{1}{t},a+\frac{1}{t}\right)$. Since a is the only point that is always in the range, it determines every property of the objects directed by it. Hence, it amounts to saying that the range is $\{a\}$.

e. At time t, $Range(L_a\cap R_b)=\left(a-\frac{1}{t},a\right)\cap\left(b,b+\frac{1}{t}\right)$. If $a<b$ then the range is $\emptyset$. If $a=b$, clearly it is $\{a\}$. If $b<a$ then if t is large enough, then $b<b+\frac{1}{t}<a-\frac{1}{t}<a$. But $\left(a-\frac{1}{t},a\right)\cap\left(b,b+\frac{1}{t}\right)=\left(b+\frac{1}{t},a-\frac{1}{t}\right)$, so that (b,a) is the range.

f. Since $Range(R_{-\infty})$ and $Range(L_\infty)$ are $(-\infty,0)$ and $(0,\infty)$ respectively, then $Range(R_{-\infty}+L_\infty)=R$.

3.2. Constructed Derivatives and Integrals

In calculus, cf. [4], the main idea of the derivative of f at a is the quotient of the form $f'_{x,y}(a)=\frac{f(x)-f(y)}{x-y}$, where $x<a<y$. For $d>0$ let $f'_d(a)=\{f'_{x,y}\,\big|\,|x-y|=d\}$. Therefore, the definition $f'(a)=\lim_{x\to y}f'_{x,y}(a)$ may be described with d replaced by R_0, obtaining:

Definition 5: Constructed derivative. Let f be a function and a is in the interior of its domain. The constructed derivative of f at a is $f^*(a)=f'_{R_0}(a)$, where $f'_d(a)=var\{f'_{x,y}\,\big|\,|x-y|=d\}$.

Theorem 6: Constructed derivative extends ordinary derivative. If f is a function differentiable in the ordinary sense, then $f^*(a)=f'(a)$.

Proof: Let f be a function differentiable at a point a. Given $\varepsilon>0$, there is $\delta>0$ so that $\left|\frac{f(x)-f(y)}{x-y}-f'(a)\right|<\varepsilon$, provided that $|x-y|<\delta$. Thus $f'_\delta(a)\subseteq\left(f'(a)-\varepsilon,f'(a)+\varepsilon\right)$. Since ε is arbitrary, then $f^*(a)=f(a)$, since the latter is always contained in $\left(f'(a)-\varepsilon,f'(a)+\varepsilon\right)$.

Some usual theorems about derivatives in Calculus may have to be modified. For example, the following:

Theorem 7: The chain rule. Suppose $f\colon B\to C$ and $g\colon A\to B$. Then, $(f\circ g)^*(a)=(f^*\circ g)(a)g^*(a)$.

When g is not constant at a neighborhood of a, otherwise the left-hand side must be calculated directly.

Proof: From Definition 5, we obtain:

$$(f\circ g)^*(a)=(f\circ g)'_{R_0}(a)$$
$$=var\{(f\circ g)'_{x,y}(a)\,\big|\,|x-y|=R_0\}$$
$$=var\left\{\frac{(f\circ g)(x)-(f\circ g)(y)}{x-y}\,\middle|\,|x-y|=R_0\right\}.$$

But when $g(x)-g(y)\neq 0$,

$$\frac{(f\circ g)(x)-(f\circ g)(y)}{x-y}$$
$$=\frac{(f\circ g)(x)-(f\circ g)(y)}{g(x)-g(y)}\cdot\frac{g(x)-g(y)}{x-y}$$
$$=f'_{R_0}(g(a))\cdot g'_{R_0}(a).$$

If $g(x)=g(0)$ it is undefined in the above line of derivation, while calculating directly from the definition, $(f\circ g)^*(a)$ is still defined.

Some theorems, however, retain their original form. For example:

Theorem 8: Multiplication rule. Let $f,g\colon A\to R$ be two functions. Then, $(fg)^*(a)=f^*(a)g(a)+f(a)g^*(a)$.

Proof: From Definition 5, we obtain:

$$(fg)^*(a)=(fg)'_{R_0}(a)$$
$$=var\{(fg)'_{x,y}(a)\,\big|\,|x-y|=R_0\}$$
$$=var\left\{\frac{(fg)(x)-(fg)(y)}{x-y}\,\middle|\,|x-y|=R_0\right\}.$$

However, the following holds:

$$\frac{(fg)(x)-(fg)(y)}{x-y}$$
$$=\frac{f(x)g(x)-f(y)g(x)+f(y)g(x)-f(y)g(y)}{x-y}$$
$$=\frac{f(x)-f(y)}{x-y}g(x)+f(y)\frac{g(x)-g(y)}{x-y}$$
$$=f'_{R_0}(a)g(a)+f(a)g'_{R_0}(a).$$

To talk about Riemann integration, cf. [4], it is indispensable to precede it with the definitions of partition and its mesh, as well as the choice function for each subinterval resulting from the partition.

Definition 6: Mesh of a partition. Let (a,b) be a real interval, where $a=-\infty$ or $b=\infty$ are also allowed. A partition of I is a finite sequence $P=(x_1,x_2,\cdots,x_n)$ so that $a=x_1<x_2<\cdots<x_n=b$. The mesh of P is the number $\max\{x_k-x_{k-1}|k=2,3,\cdots,n\}$. The mesh of the partition P is written as $|P|$.

The main idea behind the Riemann integral of f over the interval $[a,b]$ is the sum of the form $S(f,P,d,\sigma)=\sum_{k=1}^{n}f(x_k^*)(x_{k+1}-x_k)$ for $x_i\in P$, where P is a partition of $[a,b]$ with $|P|=d$ and σ is called a choice function for P, fulfilling $\sigma([x_k,x_{k+1}])=x_k^*\in[x_k,x_{k+1}]$ for $k=1,2,\cdots,n-1$.

Thus, the Riemann integral can be constructed as follows:

Definition 7: Constructed Riemann integral. Let f be a function defined on the closed interval $[a,b]$. For any natural number n, let

$w(n)=\left\{S(f,P,\sigma)|\ \sigma\ is\ a\ choice\ function\ for\ P,|P|=\frac{1}{n}\right\}.$

Let $u(x)=w(\lceil x\rceil),\ for\ x>0$. Let $s(x)$ be a variable ranging over $u(x)$. The constructed Riemann integral is given by $\int_a^b*f(x)dx\overset{\text{def}}{=}u(L_\infty)$.

Theorem 9: Constructed integrals extend ordinary integrals. If f is a function that is integrable in the ordinary sense, then $\int_a^b * f(x)dx = \int_a^b f(x)dx$.

Proof: Suppose f is a function with $\int_a^b f(x)dx$ exists. Given $\varepsilon > 0$ there is $\delta > 0$ so that for any partition P with mesh less than δ, and any σ, $\left|S(f, P, \sigma) - \int_a^b f(x)dx\right| < \varepsilon$. Then any member $q \in u(n)$ with $\frac{1}{n} < \delta$ satisfies $\left|q - \int_a^b f(x)dx\right| < \varepsilon$. Hence, we can conclude that $u(L_\infty) = \int_a^b f(x)dx$ since the latter is always contained in $(q - \varepsilon, q + \varepsilon)$, where $q \in u(n)$ where $n > \frac{1}{\delta}$.

One of the immediately emerging theorems about constructed integration is as follows.

Theorem 10: Integration by parts looks like. Let $f, g: I \to R$ where I is an interval. Then, $\int_I * f dg = \int_I * (fg)^* dx - \int_I * g df$, where $df = f^* dx$ and $dg = g^* dx$

Proof: By Theorem 8, $\int_I * (fg)^*(x)dx = \int_I * \left(f^*(x)g(x) + f(x)g^*(x)\right)dx =$
$\int_I * f^*(x)g(x)dx + \int_I * f(x)g^*(x)dx$.

Therefore, $\int_I * f(x)g^*(x)dx = \int_I * (fg)^*(x)dx - \int_I * f^*(x)g(x)dx$.

Among what is not clear yet is the relationship between $\int_I * (fg)^*(x)dx$ and fg.

For multivariate calculus, see [5]. It can be used to define a multivariate version of constructed Riemann integrals.

Directed set construction might be applied to other types of integrals, such as those discussed in [6]. For another concept which is reminiscent of integrals called *an inteduct* studied in [7]. Of course, the directed set approach can generalize interacts as well.

3.3. Natural Numbers and Power Sets

Returning to the set theoretic world, which begins with finite sets of natural numbers discussed in the Introduction, we now aim at the Cantor Theorem in the hope of proving that constructions with finite-set-valued functions are at least as good as the set universe of set theory. The Cantor Theorem states that any set can never be put into one-to-one correspondence with its power set. Now we prove why a certain construction with finite sets forms an object equivalent to the power set of natural numbers.

Definition 8: Standard part of a constructed set, super-standard part of a constructed subset.

A. Let $f: L_\infty \to U$ be a function where U is a family of sets. Then, the standard part of f is $\sigma(f) = \{u \in f | u \text{ is eventually costant}\}$.

B. For any set X, let $P(X)$ denote the power set of X, that is $\{Y | Y \subseteq X\}$.

C. Let $f: L_\infty \to \{P(S)\}_{S \in U}$ be a function where U is a family of sets, so that $f(t) = P(S)$ for some $S \in U$. Then, the super-standard part of f is $\mu(f) = \{u \in f | u \subseteq \sigma(S)\}$.

The theorem above states that the finite set valued function version of the power set can be interpreted as equal to the normal power set of N when only standard elements are concerned.

Theorem 11: Let $S(t) = [\lceil t \rceil] \stackrel{\text{def}}{=} \{1,2,\cdots,\lceil t \rceil\}$ for $t > 0$. Let the power set function $P(S): L_\infty \to \{P(S(t))\}_{t>0}$ be defined as $P(S)(t) \stackrel{\text{def}}{=} P(S(t))$ for $t > 0$. Then $\mu(P(S)) = P(N)$, where $N = \{1,2,\cdots\}$.

Proof: Suppose that $X \in \mu(P(S))$. Then $X \subseteq \sigma(S)$. Hence, $X \subseteq N$. So, $\mu(P(S)) \subseteq P(N)$. Conversely, let $M \in P(N)$. Then $M \subseteq N$. Let $M^* = \{m^* | m \in M\}$ where $m^*(t) = m$ for $t > 0$. But then $M^* \in \mu(P(S))$. So, $P(N) \subseteq \mu(P(S))$, establishing $P(N) = \mu(P(N))$.

Using the same reasoning, k times of applying power set operation to the constructed version of N gives a constructed version of $P^k(N)$. Now, we reason that a certain construction with finite sets is equivalent to the infinite-fold power set of natural numbers. The definition below assumes Zermelo–Fraenkel, namely the ZF set theory, especially the Axiom of Regularity. As a digression, for an account of Russell's paradox, see [8].

Definition 9: Consistent elements; consistent part of infinite power set; equality of constructed elements and elements of the infinite power set. All the sequences below are the sequential notation of Definition 2 for functions on R^+.

a. An element $U \in \prod_{k=1}^n P^k(N) \stackrel{\text{def}}{=} P(N) \times PP(N) \times \cdots \times P^n(N)$ is called consistent f $U_{k-1} \in U_k$ for all $k = 1,2,\cdots n$. Likewise, $U \in \prod_{k=1}^\infty P^k(N)$ is called consistent if $U_{k-1} \in U_k$ for all $k = 1,2,\cdots n$.

b. Suppose $X = \prod_{k=1}^\infty P^k(N)$. Then, the consistent part of X is $\kappa(X) \stackrel{\text{def}}{=} \{U \in X | U \text{ is consistent}\}$.

c. Suppose $U \in \prod_{k=1}^\infty P^k(N)$ and $V \in P^\infty(N)$. Then by Axiom of Regularity in *ZF*, there is an integer k *so that* $V = V_\infty = V_k \ni V_{k-1} \ni \cdots \ni V_0$, for some V_0 that has no element further, namely $\emptyset$ or an atom. U is said to be equal to V if there are $U_1 \in P(N), U_2 \in P^2(N), \cdots$ with $V_0 = U_0 \in \sigma(U_1) = V_1 \in \sigma(U_2) = V_2 \in \cdots \in V$.

Now, we are ready to prove that

Theorem 12: $\kappa\left(\prod_{k=1}^\infty P^k(N)\right) = P^\infty(N)$, where $P^\infty(N) = \cdots P(P(P(N)$.

Proof: Suppose $U \in \kappa\left(\prod_{k=1}^\infty P^k(N)\right)$. Since U is consistent, then $(U_1, U_2, \cdots)$ corresponds to the development of an element of $P^\infty(N)$. To see this, assume that no element of $P^\infty(N)$ has development that corresponds to U. Then, there is an integer n so that $U(n)$ does not correspond to any element of $P^\infty(N)$. But $U(n-1) \in U(n)$ so that $U(n) = \{U_{n-1}\} \cup X$ for some $X \subseteq P^n(N)$, while for any $Y \subseteq P^n(N)$, $Y \cup \{U_{n-1}\}$ is a legitimate candidate for U_n. A contradictory situation. Therefore, it must be the case that U corresponds to the development of an element V of $P^\infty(N)$. Hence $U = V$. Thus $\kappa\left(\prod_{k=1}^\infty P^k(N)\right) \subseteq P^\infty(N)$.

Conversely, assume $W \in P^{\infty}(N)$. Then, by the Axiom of Regularity, there is an integer k so that $W = W_{\infty} = W_k \ni W_{k-1} \ni \cdots \ni W_0$, for some W_0 that has no element. Then starting from W_0, there is W_1 so that $W_0 \in W_1$ and W_1 leads to the development of W at stage 1. Continuing, there is W_2 so that $W_1 \in W_2$ and W_2 leads to the development of W at stage 2. and so on. Thus $(W_1, W_2, \cdots)$ is an element of $\kappa\left(\prod_{k=1}^{\infty} P^k(N)\right)$ that is regarded equal to W. Therefore, $\kappa\left(\prod_{k=1}^{\infty} P^k(N)\right) \supseteq P^{\infty}(N)$, establishing $\kappa\left(\prod_{k=1}^{\infty} P^k(N)\right) = P^{\infty}(N)$.

The theorem above states that $P^{\infty}(N)$ can still be accommodated by a construction with finite sets.

The next section presents further discussions.

4. Results and Discussion

This section considers the consequences, examples, and applications of what was developed in the last section. Our results on numbers are comparable to surreal numbers [9]. [10] presents an interesting account of the structure of surreal number birthdays.

4.1. Limits of Functions, Constructed Derivatives and Integrals, and Sums of Infinite Series

The constructed derivative of functions in Definition 5 always exists, in contrast to the ordinary derivative. It may be helpful because it distinguishes different cases of failure of ordinary derivatives.

Example 1: At $x = 0$, the function

$$f(x) = \begin{cases} 1, if\ x\ is\ rational, \\ 0, otherwise, \end{cases}$$

has no derivative in the ordinary sense. For $d > 0$ where $d = x - y$,

$$\frac{f(x)-f(y)}{x-y} = \begin{cases} 0, & if\ x, y \in Q\ or\ x, y \notin Q, \\ \frac{1}{d}, & if\ x \in Q, y \notin Q, \\ -\frac{1}{d}, & if\ x \notin Q, y \in Q. \end{cases}$$

Thus $f^*(0) = var\left(\bigcup_{d>0}\left\{0, \frac{1}{d}, -\frac{1}{d}\right\}\right)$. Hence, $f^*(0) = L_{\infty} \cup 0 \cup R_{-\infty}$.

We may interpret that the slope of the function has three components, one being horizontal, another being vertically upward, and the last one being vertically downward.

Similar to its derivative counterpart, the constructed Riemann integral of Definition 7 gives values to any choice of function, in contrast to the ordinary Riemann integral. Again, similar to the constructed derivative, it may be helpful to study different cases of failure, as in [3].

Example 2: Let f be example's function 2. Then f has no Riemann integral in the ordinary sense. Let n be a natural number. Then, $\sum_{k=1}^{n-1} f(x_k^*)(x_{k+1} - x_k) = var([0, b-a])$ because for any $c \in [0, b-a]$, if we let $x_1 = a < x_2 = a + c < x_3 < \cdots < x_n = b$, and x_1 rational, while x_k irrational for the rest of k's, then $\sum_{k=1}^{n-1} f(x_k^*)(x_{k+1} - x_k) = c$.

Therefore, referring to Definition 3.2, $w(n) = [0, b-a]$. Therefore, $\int_a^b * f(x)dx$ as a variable has range $[0, b-a]$.

Example 3: The so-called parametric integration in [3] is also an example that may be undefined in the ordinary sense. For example, $\int_0^{\infty} \sin x\, dx$ is undefined, but $\int_0^{\infty} * \sin x\, dx = -\cos(L_{\infty})$.

Considering functions of the form $f: R^* \to R^*$ with $R^* \overset{\text{def}}{=} \{R_{-\infty}\} \cup \{R_{-a}, a, L_a\}_{a \in R} \cup \{L_{\infty}\}$, there are some interesting facts such as the following:

Theorem 13: Let $f: I \to R$ be a real function where I is an interval $[a, b]$. For any $K \in R \cup \{\infty, -\infty\}$, there is an expansion $f^*: R^* \to R^*$ of f so that $\int_a^b * f^*(x)dx = K$ regardless of the value of $\int_a^b f(x)dx$.

Proof: Suppose $c \in I$, $M = \int_a^b f(x)dx$, and

$$f^*(x) = \begin{cases} f(x), if\ x \in I, \\ \dfrac{-(M-K^*)}{R_c - c}, if\ x = dR_c, where\ d \in (0,1), \\ 0, elsewhere, \end{cases}$$

with $K^* = \begin{cases} K, if -\infty < K < \infty, \\ L_{\infty}, \quad if\ K = \infty, \\ R_{-\infty}, \quad if\ K = -\infty. \end{cases}$

Then

$$\int_a^b * f^*(x)dx = \int_a^b * f(x)dx + \int_c^{R_c} * f^*(x)dx$$
$$= \int_a^b f(x)dx + \int_c^{R_c} * \left(\frac{-(M-K^*)}{R_c - c}\right) dx$$
$$= M - (M - K^*) = K^* = K.$$

Closely related to calculus is the issue of the sums of series. Definition 2 gives the functional equivalents of the sums of convergent series. The following are two examples.

Example 4:

a. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \left(\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k}\right)_{n=1}^{n=\infty} = \left(1, \frac{1}{2}, \frac{5}{6}, \frac{2}{3}, \cdots\right)$

b. $\sum_{n=1}^{\infty}\frac{1}{n^2} = \left(\sum_{k=1}^{n}\frac{1}{k^2}\right)_{n=1}^{n=\infty} = \left(1, \frac{5}{4}, \frac{49}{36}, \frac{405}{238}, \cdots\right)$.

However, Definition 2 also gives the values of the sums of arbitrary series. This does not eliminate the importance of convergence issues because they are crucial in determining which sums are convertible to real numbers and which ones are to be left as non-standard numbers. The following example gives two well-known divergent series. For the number theoretic and probability distribution approach to sums of series, see [11].

Example 5:

a. $\sum_{n=1}^{\infty}\frac{1}{n} = \left(\sum_{k=1}^{n}\frac{1}{k}\right)_{n=1}^{n=\infty} = \left(1, \frac{3}{2}, \frac{11}{6}, \frac{50}{24}, \cdots\right)$.

b. $\sum_{n=1}^{\infty}(-1)^n = \left(\sum_{k=1}^{n}(-1)^k\right)_{n=1}^{n=\infty} = (-1, 0, -1, 0, \cdots) = 0 \cup -1$.

For other approaches to the sums of *divergent* series, see [12].

4.2. Universe of Sets and Ordinal-Like Constructions

The following theorem is derived almost verbatim as in Theorems 8 and 9 by replacing power set operation $P(X)$ with $V(X) \stackrel{\text{def}}{=} X \cup P(X)$. Hence, the proof is omitted.

Theorem 14: Let S be a set definable as a standard part of a finite set construction with a directed set. Then so is $V(S)$ and $V^\infty(S)$.

The von Neumann Universe is the union $V = \bigcup_\alpha V_\alpha$ where α runs over all ordinal numbers. Here $V_0 = \emptyset$, $V_{\beta+1} = V_\beta \cup P(V_\beta)$, and $V_\lambda = \bigcup_{\mu<\lambda} V_\mu$ if λ is a limit ordinal. V in Set Theory can be imitated in the lines of Theorems 8 and 9. The proof below assumes that "Every infinite set is a countable union of smaller sets" $\cdots (+)$

[13] proves that in his model, this assumption holds.

Theorem 15: For any ordinal number α, the α-th von Neumann universe V_α is definable as a construction with finite sets.

Proof: $V_0 = \emptyset$ clearly corresponds to the constant directed set $var(\emptyset)$. Suppose $\alpha = \beta + 1$ and V_β satisfies the theorem. Then $V_\alpha = V_\beta \cup P(V_\beta)$. By Theorem 11, there are constructed versions V_β^* and $P(V_\beta)^*$ of V_β and $P(V_\beta)$ respectively. Hence, there is a constructed version V_α^* of their union V_α. Suppose α is a limit ordinal and that any subset of V_α with smaller cardinality has a constructed version. By (+), $V_\alpha = \bigcup_{n=1}^\infty W_n$ where W_n is of smaller cardinality than V_α. Suppose W_n^* is a constructed version of W_n. Then, employing Definition 2, $\bigcup_{n=1}^\infty W_n^*$ is a constructed version of V_α.

From Theorem 15, we know that the constructed universe of finite sets is as good as the von Neumann universe, albeit under a certain assumption (+).

Lastly, let us take a glimpse at ordinal-like construction. To mimic ordinal numbers in set theory, one considers additions of 1 to 0 which may be infinitely many times. In conformity with Definition 2, $o \stackrel{\text{def}}{=} 1 + 1 + 1 + \cdots$ is produced by the number-sequence $(1,1,1,\cdots)$ and the operation-sequence $(+,+,+,\cdots)$, namely $(1, 1+1, 1+1+1, \cdots) = (1,2,3,\cdots)$, where the sequence notation denotes the real function $o(x) = \lceil x \rceil$.

The process can be continued indefinitely in the following manner:

$o + 1 = (1,2,3,\cdots) + (1,1,1,\cdots) = (2,3,4,\cdots)$,

$o + 1 = (2,3,4,\cdots) + (1,1,1,\cdots) = (3,4,5,\cdots)$,

and so on.

$2o = 2 \cdot (1,2,3,\cdots) = (2,4,6,\cdots) = o + o$,

$3o = 3 \cdot (1,2,3,\cdots) = (3,6,9,\cdots)$,

and so on.

$o^2 = o \cdot o = (1,2,3,\cdots) \cdot (1,2,3,\cdots) = (1,4,9,\cdots)$,

$o^3 = (o \cdot o) \cdot o = \big((1,2,3,\cdots) \cdot (1,2,3,\cdots)\big) \cdot (1,2,3,\cdots) = (1^3, 2^3, 3^3, \cdots)(1,8,27,\cdots)$,

and so on.

$o^o = (1,2,3,\cdots)^{(1,2,3,\cdots)} = (1^1, 2^2, 3^3, \cdots)$,

$o^{o^o} = \left(1^{1^1}, 2^{2^2}, 3^{3^3}, \cdots\right)$, and so on.

The Cantor Normal Form yields that every ordinal number has the form $\omega^{\beta_n} + \omega^{\beta_{n-1}} + \cdots + \omega^{\beta_0} \cdots (*)$ for some ordinals $\beta_n \geq \beta_n \geq \cdots \geq \beta_0$. It is natural to conjecture that the ordinal $(*)$ is expressible n in the form $o^{b_n} + o^{b_{n-1}} + \cdots + o^{b_0} \cdots (**)$, where $b_n \geq b_{n-1} \leq \cdots \leq b_0$ are already obtained.

Therefore, we can at least propose a conjecture that there is 1-1 correspondence between the ordinals and their constructed versions, namely $(*)$ and $(**)$. They cannot be isomorphic because unlike the constructed versions, addition and multiplication of ordinals are not always in conformity with the field axioms. However, Theorem 15 indirectly proves this conjecture under assumption (+).

One advantage of constructed ordinal numbers over ordinal numbers is their operability to common functions previously known only to real numbers. Since operations on time functions are carried out term wise, any operation on ordinal numbers is performed similarly. For example, $\sqrt{\omega} = \left(\sqrt{1}, \sqrt{2}, \sqrt{3}, \cdots\right)$, $\sin\omega = (\sin 1, \sin 2, \sin 3)$, and so on.

As a comparative study, for an obvious development of ordinals, cf. [14], for the development of real numbers from sets, cf. [15] , and for overall set theory, cf. [16].

5. Conclusion

Using directed sets, nested directions, namely the ones eventually becoming ever smaller, capture specific non-standard individual objects such as infinitesimal and infinite numbers, sums of infinitely many operations such as sums of series or ordinal-like numbers. While expanding directions, namely the ones eventually becoming ever more, capture comprehensive collections of objects such as infinite sets from finite sets, even their power sets, or infinitely many times of the power set operation.

The derivative of functions can be constructed with parameters for the distance of two points where the slope is measured. The definition coincides with the usual derivative when the latter is defined, whereas in other cases, it may end up in non-standard numbers.

The Riemann integral of functions over an interval can use parameters for the mesh of partition that defines the approximating sum. The definition coincides with the usual Riemann integral when defining the latter, whereas in other cases, it may end up in non-standard numbers.

The constructed calculus modifies some theorems and also retains some of them. However, the standard part is still exactly as it is in classical calculus.

Ordinal numbers are constructed through a definition of infinitely many applications of numerical operations, namely the repeated addition of the number 1, where indefinitely many times is allowed. However, the resulting objects are not exactly ordinal numbers because the former is commutative; it is possible to assume that they have one-to-one correspondence.

There is a construction with finite sets that produces the set of all-natural numbers when considering only standard numbers. The power set of this set is also reproducible as a constructed object, similarly with infinitely many applications of power set operation to this set. The universe of a set is within the universe of constructed sets, but it remains a conjecture expressed as one.

The paper contributes to introducing constructions using directed sets and applying them to fill the gaps left by conventional mathematics. After filling the gaps, they are subject to further study. In addition, the result states that the universe of directed sets of finite sets is as good as the von Neumann universe, which also entails a number system corresponding to ordinal numbers but retaining field properties.

Hence, we hope this paper will further the studies of some undefined notions in calculus. In addition, it is possible to approximate a little closer to light that mathematics can express (almost) all within the Cermelo−Frenkel universe as finite sequences of finite structures. Hopefully, the directed set approach can still be simplified further in the future to provide more benefits.

References

[1] GOLDBRING I., and WALSH S. An Invitation to Nonstandard Analysis and Its Recent Applications. *Notices of the American Mathematical Society*, 2019, 66(6): 842-851.
[2] SAIGO H., and NOHMI J. Categorical Nonstandard Analysis. *Symmetry*, 2021, 13(9), 1573. https://doi.org/10.3390/sym13091573
[3] SALIM R.A., and ERNASTUTI. The Definability and Strength of Fourier Series Transform Obtained from Employing Parametric Integration. *Journal of Hunan University (Natural Sciences)*, 2023, 50(8): 160-167. https://doi.org/10.55463/issn.1674-2974.50.8.13
[4] BARTLE R.G., and SHERBERT D.R. *Introduction to Real Analysis*. 4th ed. John Wiley & Sons, Hoboken, New Jersey, 2011.
[5] MUNKRES J.R. *Analysis on Manifolds*. Addison Wesley Publishing Company, Redwood City, CA, 1991.
[6] GONZALEZ-VELASCO E.A. The Lebesgue Integral as a Riemann Integral. *International Journal of Mathematics and Mathematical Sciences*, 1987, 10(4): 693-706. DOI: 10.1155/S0161171287000802
[7] GRABINSKI M., and KLINKOVA G. Like a Sum Is Generalized into an Integral, a Product May Be Generalized into an Inteduct. *Applied Mathematics*, 2023, 14: 279-289. DOI: 10.4236/am.2023.145017
[8] SALIN H. *On Russell's Paradox and Attempted Resolutions*. Bachelor's Thesis. Department of Historical, Philosophical and Religious Studies, Umea University, Umea, Sweden, 2023.
[9] DACK T.-J.S., and TRESSL M. Elementary *Theory of Surreal Numbers*. Master's Thesis. University of Manchester, Manchester, 2018.
[10] ROUGHAN M. Surreal Birthdays and Their Arithmetic. *Mathematics Magazine*, 2023, 96(3): 329-343. https://doi.org/10.1080/0025570X.2023.2205819
[11] KUPERBERG V. Sums of Singular Series with Large Sets and the Tail of The Distribution of Primes. *The Quarterly Journal of Mathematics*, 2023, Advance articles. https://doi.org/10.1093/qmath/haad030
[12] CHAGAS J.Q., TENREIRO MACHADO J.A., and LOPES A.M. Overview in Summabilities: Summation Methods for Divergent Series, Ramanujan Summation and Fractional Finite Sums. *Mathematics*, 2021, 9(22), 2963. https://doi.org/10.3390/math9222963
[13] GITIK M. All Uncountable Cardinals can be Singular. *Israel Journal of Mathematics*, 1980, 35(September): 61-88. https://doi.org/10.1007/BF02760939
[14] VAN DER BERG I., and JANELIDZE Z. *An Axiomatic Approach to the Ordinal Number System*. Stellenbosch University, Matieland, South Africa, 2021.
[15] ENDERTON H.B. *Elements of Set Theory*. Academic Press, New York, 1977.
[16] FRAENKEL A.A., BAR-HILLEL Y., and LEVY A. F*oundations of Set Theory*. Elsevier, Amsterdam, 1973.

参考文:

[1] GOLDBRING I. 和 WALSH S. 非標準分析及其近期應用的邀請。美國數學會通知，2019，66(6)：842-851。
[2] SAIGO H. 和 NOHMI J. 分類非標準分析。對稱性，2021, 13(9), 1573。https://doi.org/10.3390/sym13091573
[3] SALIM R.A. 和 ERNASTUTI。採用參數積分所獲得的傅立葉級數轉換的可定義性和強度。湖南大学学报（自然科学版）, 2023, 50(8): 160-167. https://doi.org/10.55463/issn.1674-2974.50.8.13
[4] BARTLE R.G. 和 SHERBERT D.R. 實分析簡介。第四版。約翰威利父子公司，新澤西州霍博肯，2011。
[5] MUNKRES J.R. 流形分析。艾迪生韋斯利出版公司，加州雷德伍德城，1991年。
[6] GONZALEZ-VELASCO E.A. 勒貝格積分作為黎曼積分。國際數學與數學科學雜誌，1987， 10(4): 693-706。DOI：10.1155/S0161171287000802
[7] GRABINSKI M. 和 KLINKOVA G. 就像將和推廣為積分一樣，乘積也可以推廣為積分。應用數學，2023，14：279-289。DOI：10.4236/am.2023.145017
[8] SALIN H. 論羅素悖論和嘗試的解決方案。學士論文。於默奧大學

歷史、哲學和宗教研究系，瑞典於默奧，2023年。

[9] DACK T.-J.S. 和 TRESSSL M. 超現實數基本理論。碩士論文。曼徹斯特大學，曼徹斯特，2018。

[10] ROUGHAN M. 超現實的生日及其算術。數學雜誌，2023，96(3)：329-343。https://doi.org/10.1080/0025570X.2023.2205819

[11] KUPERBERG V. 大集奇異級數和質數分佈尾部。《數學季刊》，2023，高級文章。https://doi.org/10.1093/qmath/haad030

[12] CHAGAS J.Q.、TENREIRO MACHADO J.A. 和 LOPES A.M. 可和性概述：發散級數的求和方法、拉馬努金求和和分數有限和。數學，2021, 9(22), 2963。https://doi.org/10.3390/math9222963

[13] GITIK M. 所有不可數基數都可以是單數。以色列數學雜誌，1980，35（九月）：61-88。https://doi.org/10.1007/BF02760939

[14] VAN DER BERG I. 和 JANELIDZE Z. 序數系統的公理方法。斯泰倫博斯大學，南非馬蒂蘭，2021。

[15] ENDERTON H.B. 集合論的要素。學術出版社，紐約，1977年。

[16] FRANKEL A.A.、BAR-HILLEL Y. 和 LEVY A. 集合論基礎。愛思唯爾，阿姆斯特丹，1973年。