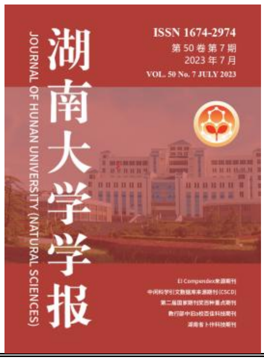

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Mathematical Modeling Using Second-Order Differential Equations: A Case Study of a Mass-Spring System

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Abstract: This study aims to solve a second-order ordinary differential equation that mathematically models the position of a mass attached to a spring over time, with a specific focus on the two-second time interval. This mass-spring system is a physical phenomenon that exhibits behavior closely resembling a sinusoidal wave. This study is conducted within the context of an engineering-oriented differential equations course at a Colombian university using four strategies to accomplish this objective: 1) theoretical modeling, 2) simulated modeling, 3) mathematical modeling by solving a first-order differential equation, and 4) mathematical modeling by solving a second-order differential equation. The analysis of the results will primarily involve comparing the coefficient of determination 2^2 with the response obtained from the theoretical model.

Keywords: numerical differentiation, ordinary differential equations, mathematical modeling, mass-spring system.

使用二階微分方程進行數學建模：質量彈簧系統案例研究

摘要：本研究旨在求解一個二階常微分方程，該方程對附著在彈簧上的質量隨時間的位置進行數學建模，特別關注兩秒的時間間隔。這種質量彈簧系統是一種物理現象，表現出與正弦波非常相似的行為。這項研究是在一所哥倫比亞大學以工程為導向的微分方程課程的背景下進行的，使用四種策略來實現這一目標：1）理論建模，2）模擬建模，3）通過求解一階微分方程進行數學建模，4)通過求解二階微分方程進行數學建模。結果分析主要涉及將決定係數R^2 與從理論模型獲得的響應進行比較。

关键词：數值微分、常微分方程、數學建模、質量彈簧系統。

1. Introduction

Scientific and technological advancements in recent times have shown exponential growth in academic developments, contributing to improving our way of life through engineering. Therefore, education plays a crucial role in training future engineers within the university setting.

In this context, teaching university-level mathematics, especially the course on Differential Equations, introduces a series of ideas and implements new teaching and research methods through mathematical modeling that, in turn, leads to new ways of life. Various forms of mathematical modeling of phenomena and/or processes are used to refine new concepts and developments [1].

By incorporating these contributions into the curriculum of the Differential Equations course, students can develop a closer relationship with mathematics, enabling better assimilation of mathematical concepts and generating new ideas in problem-solving processes [2].

Differential equations facilitate the interaction and integration of knowledge between fundamental mathematics and engineering applications. They allow for the investigation of various everyday problems by approximately describing the behavior of real-world situations through mathematical models [3].

It has been demonstrated that mathematical modeling identified by differential equations corresponds to an initial phase in the modeling process known as the "Mathematical Domain." These models are commonly used in different courses, such as basic mathematics, differential and integral calculus, statistics, and more [4].

In textbooks typically used in differential equations courses, the problems and exercises often provide a mathematical model, frequently referred to as the initial value problem. Among these, the mass-spring model (also known as Hooke's Law) without external forces stands out, as presented in [5].

This system consists of a mass m hanging from a rigid spring with an elasticity constant r. Here, gravity causes the mass to move downward, while the restoring force of the spring acts in the opposite direction, upward, as shown in Figs. 1 and 2.

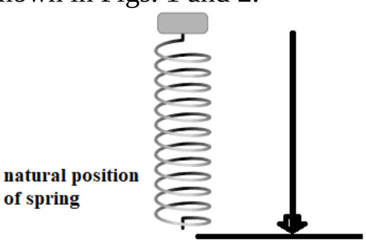

Fig. 1 The natural position of the spring

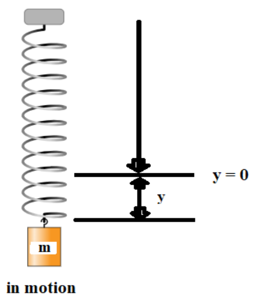

Fig. 2 Spring with an attached mass m oscillating back and forth

Upon observing Fig. 2, if the mass m deviates from the equilibrium position, it will oscillate up and down. This behavior can be modeled using a second-order ordinary differential equation with the following constant coefficients:

$$\begin{cases} my'' + \gamma y' + ry = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y'_0 \end{cases} \quad (1)$$

Here, m represents the mass hanging from the spring, γ is the damping constant, r is the spring constant, t_0 is the initial time, y_0 and y_0' correspond to the initial position and velocity, respectively.

The differential equation (1) is classified as an ordinary, homogeneous, linear, second-order equation with constant coefficients. It has two linearly independent solutions of the form $y = f(t)$ and the unique property that its second derivative can be expressed as a linear combination of its first derivative and the function itself [6].

Drawing upon research [2], particularly on Newton's cooling and/or heating law, and leveraging numerical differentiation using centered finite differences with five steps of the first order for each time instant (t_i), as demonstrated in [8], it can be shown that for a table of data with two equally spaced variables with respect to the independent variable, we have:

$$f'(t_i) = \frac{-f(t_{i+2}) + 8f(t_{i+1}) - 8f(t_{i-1}) + f(t_{i-2})}{12\square} \quad (2)$$

All the above-mentioned considerations stimulated the creation of a strategy to model the mass-spring system using numerical methods. In this approach, centered finite divided differences with five steps are employed to obtain the numerical evaluation of the second derivative. It is assumed that the error is on the order of o($\square^4$) for each time instant (t_i), as described in [8]. This aspect adds novelty to the methodology.

$$f''(t_i) = \frac{-f(t_{i+2}) + 16f(t_{i+1}) - 30f(t_i) + 16f(t_{i-1}) - f(t_{i-2})}{12\square^2}, \quad (3)$$

where $\square$ represents the difference between two consecutive data points or the step size, i.e., $\square = t_{i+1} - t_i$, $f(t_i)$ denotes the evaluation of f at time t_i. It is crucial to note that the time partition adheres to the behavior described in (4), which is an integral component of the five-step partition,

$$\begin{cases} t_{i-2} = t_i - 2\square \\ t_{i-1} = t_i - \square \\ t_{i+1} = t_i + \square \\ t_{i+2} = t_i + 2\square \end{cases} \quad (4)$$

Using sinusoidal regression, a mathematical expression in the form of a sine wave is approximated. This expression effectively represents the behavior of the solution to Hooke's Law differential equation in terms of the different phases of the modeling cycle, as highlighted by [4].

The obtained mathematical expressions are evaluated alongside the observed data in the same table, and the coefficient of determination R^2 is employed for comparison. This coefficient, as indicated in the following equation, quantifies the goodness-of-fit between the modeled expressions and the observed data:

$$R^2 = \frac{\sum_{i=1}^{m}(\hat{y_i}-\bar{y})^2}{\sum_{i=1}^{m}(\hat{y_i}-\bar{y})^2+\sum_{i=1}^{m}(y_i-\hat{y_i})^2} \quad (5)$$

where (y_i) represents the value predicted by the model at x_i , and $\bar{y}$denotes the mean. This indicator measures the amount of variation in the data explained by the model [2]. This value allows us to assess how closely the obtained data align with a reference pattern in the interval $0, 1$. The closer the value tends to 1, the better the approximation of the model to the observed data.

2. Theoretical Framework

One of the few studies that address mathematical modeling and second-order differential equations was conducted by [10] as an approach to mathematics education in the context of problem-solving.

In the field of applications and modeling, particularly with second-order ordinary differential equations, there are studies such as those by [9] in the field of economics. [11] explored specific and contextual scenarios when using the mathematical model in electric vehicles.

3. Materials and Methods

This section describes the materials and methods used to achieve the research objectives. This section outlines the materials employed, the equipment utilized, and the proposed strategies for teaching the Differential Equations course.

3.1. Materials Used

- USB memory stick
- Universal support
- Springs
- Masses
- Mass support of different values

3.2. Equipment Used

- Computer
- PhET simulation (online)
- PASPORT motion sensor - PS-2103A
- Xplorer GLX - PS-2002

3.3. Teaching Strategies

The proposed teaching strategies contribute to developing a Differential Equations course for engineering students in a public university in Colombia through mathematical modeling. These strategies are based on a master's thesis [7].

Furthermore, these strategies are implemented in the chapter specifically dedicated to higher-order differential equations and their applications, as it is in this context that the mass-spring model is introduced.

The mathematical modeling strategy is synthesized by following the recommendation of Professor Ruth [4] and transitioning through the following four tools:

Strategy 1: Standard model – theoretical model;

Strategy 2: Using the PhET simulator;

Strategy 3: Mathematical modeling by solving a first-order differential equation;

Strategy 4: Mathematical modeling by solving a second-order differential equation.

In the first strategy, the differential equation given in (1) is considered, disregarding the effects of damping and external forces. This model is commonly used in differential equation courses under the pseudo-concrete domain, where variables are explicitly designated. In addition, initial value problems are provided in most exercises, and solving them often involves using methods such as undetermined coefficients.

In the second strategy, the PhET interactive simulation from the University of Colorado Boulder (https://phet.colorado.edu/es/simulation/mass-spring-lab) is a complementary tool in modeling the mass-spring system. Data obtained from the simulation are used to perform sinusoidal regression using the GeoGebra software, resulting in an expression y(t) that models the vertical displacement of the mass for any given time within the working interval.

In the third strategy, experimental data on the position of the mass at different time points are collected. Subsequently, by utilizing first-order numerical differentiation as described in (2), the solution to the initial value problem is obtained, following the recommendation of [2].

Finally, in the fourth strategy, which is the novelty of this research, the second numerical derivative is determined based on the data obtained from the same experimental situation as in strategy 3. This derivative is used to establish a second-order differential equation using sinusoidal regression:

$$y^{''(t)} = Asen(\omega t + \varphi) + D \quad (6)$$

Furthermore, through double integration, the respective solutions y(t) are obtained.

The abovementioned approach follows the recommended strategy by [4] for addressing the cycles of mathematical modeling of a phenomenon in the following manner.

In the first part, students are introduced to the transition between the stages Real Situation → Pseudo Concrete Model → Mathematical Model, using Newton's second law as the reference model.

In the second part, students are exposed to the transition between the stages Real Situation → Pseudo Concrete Model → Physical Model → Mathematical Model, using the PhET simulator.

In the third part, students undergo the same transitions as in the second part, but this time, they use the physical and technological resources of the physics laboratory and employ numerical methods for the mathematical model.

Finally, two types of comparisons are conducted. First, models 2, 3, and 4 are compared with reference model 1, and second, models 1, 2, 3, and 4 are compared with the actual data obtained in the laboratory.

4. Case Study

Next, we solve the problem of the mass-spring system by applying the four strategies to obtain a mathematical expression that relates vertical displacement to time. In this case, a mass block of 200 g (0.2 kg) elongates the spring by 27 cm (0.27 m). The mass is initially released from a resting position located 20 cm (0.2 m) below the equilibrium position, as shown in Fig. 1.

Considering Hooke's Law in the mass-spring system, the elastic force F_e is

$$F_e = ry, \quad (7)$$

where r is the spring constant, y is the vertical elongation of the spring.

The weight of the mass is:

$$W = mg, \quad (8)$$

where m represents the mass of the block attached to the spring with the acceleration of gravity g.

Combining equations (7) and (8), we can express the spring constant r

$$r = \frac{mg}{y} \quad (9)$$

Substituting the values from the experiment, we find that:

$$r = 7.26\ \frac{N}{m}.$$

Considering the absence of damping effects ($\gamma = 0$), the differential equation (1) becomes:

$$0.2y'' + 7.26y = 0, \qquad y(0) = 0.2, \qquad y'(0) = 0.$$

We proceed to solve the equation following the proposed strategies as follows.

4.1. Strategy 1

The auxiliary equation is:

$$n^2 + 36.3 = 0 \quad (10)$$

In this last expression, the value of $\sqrt{36.3} = 6.025$ corresponds to the angular frequency of the system w.

The solutions are complex conjugates given

$$n = \pm\sqrt{36.3}\, i.$$

The solution for (1) is:

$$y(t) = c_1 \cos\left(\sqrt{36.3}\, t\right) + c_2 sen\left(\sqrt{36.3}\, t\right).$$

where $y(t)$ represents the position of the mass m as a function of time t. This motion of the mass is called simple harmonic motion and has a period of motion (the time it takes to complete one oscillation) T, which is equivalent to

$$T = \frac{2\pi}{w},$$

$$T = \frac{2\pi}{\sqrt{36.3}} = 1.042\ s.$$

Applying the initial conditions of (1), $y(0) = 0.2$ and $y'(0) = 0$, we find:

$c_1 = 0.2$, and $c_2 = 0$.

Therefore, the solution to the problem described in (10) is of the following form:

$$y(t) = 0.2\cos(6.025\, t) \quad (11)$$

4.2. Strategy 2

Using the PhET simulator, as described earlier, we hang a mass of 200 g from the spring (Figure 1) and adjust it to elongate by 27 cm. The simulator includes indicators of mass in equilibrium, movable reference, gravity, and damping.

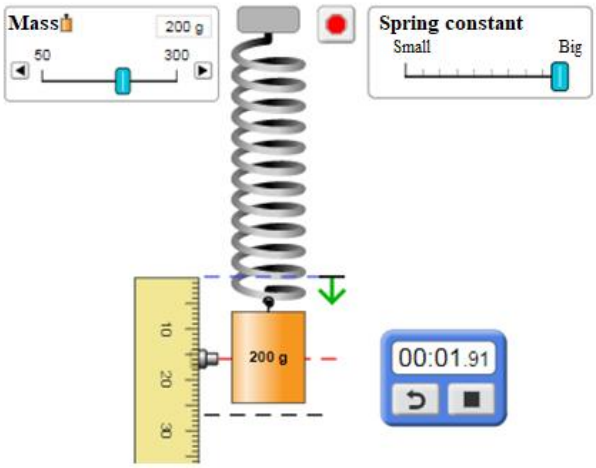

Fig. 3 Record the vertical position of the 0.2 kg mass over a time of 1.91 s

Next, the data are recorded in a GeoGebra spreadsheet to perform a sinusoidal regression (Analysis of regression with two variables) and obtain the corresponding coefficient of determination R^2. Fig. 2 shows the graphical representation and the solution.

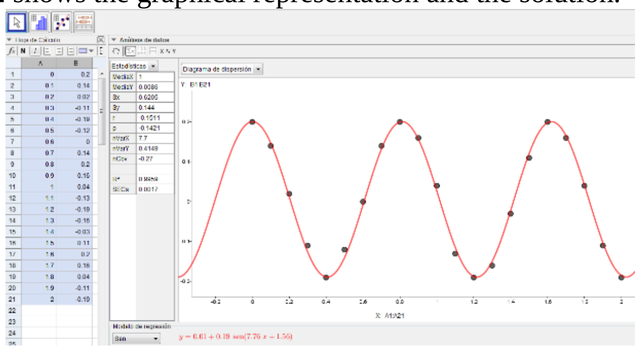

Fig. 4 Graph and sinusoidal regression in GeoGebra using data obtained from the PhET simulator

Using GeoGebra and sinusoidal regression, we obtain expression (12) with a coefficient of determination $R^2 = 0.9959$:

$$y(t) = 0.19\, sen(7.76t + 1.56) + 0.01 \quad (12)$$

4.3. Strategies 3-4

Using the same information from the previous strategies, we store the data in a recording device Xplorer GLX – PS-2002. The data are then transferred from the memory stick to a laptop using a text editor.

Fig. 5 Experimental setup for strategies 3 and 4 in the laboratory; materials and equipment used: universal support, PASPORT motion sensor (placed below the mass), and Xplorer GLX - PS-2002 reader

We apply numerical differentiation using centered divided differences with an error of order $o(\square^4)$ at five steps, obtaining the first derivative for each time point using the expression given in (2).

Considering that the instrument takes data spaced in time at intervals of 0.02 seconds, we assume that the step size is 0.02.

The data are analogously stored in a GeoGebra spreadsheet, and upon graphing, a sinusoidal regression model is obtained to represent the first-order ordinary differential equation in the form:

$$y' = f(t) = Asen(\omega t + \varphi) + D \tag{13}$$

Fig. 6 shows the graph.

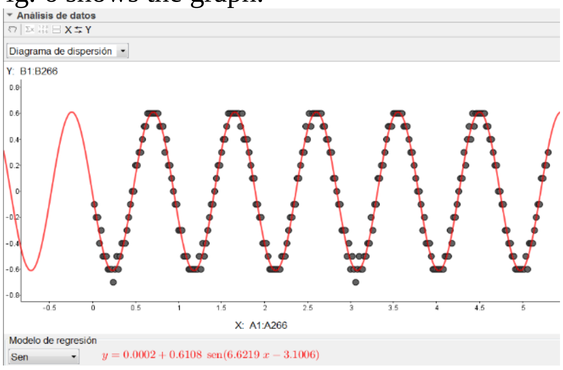

Fig. 6 Graph in GeoGebra of the first numerical differentiation and its sinusoidal regression representing a first-order differential equation

The coefficients of determination R^2 are determined for each type of regression or first-order differential equation.

Análisis de datos

Estadísticas

MediaX	2.6599
MediaY	-0.0313
Sx	1.5329
Sy	0.4314
r	0.0156
ρ	0.0141
nVarX	620.3089
nVarY	49.13
nCov	2.7151
R²	0.99
SECs	0.4935

Fig. 7 Data analysis verifying the coefficient of determination $R^2 = 0.99$ for the first numerical differentiation

Next, by solving differential equation (13) using the method of separable variables, we obtain:

$$\frac{dy}{dt} = 0.611\, sen(6.62t - 3.1) + 0.0002,$$
$$dy = (0.611\, sen(6.62t - 3.1) + 0.0002)dt,$$
$$\int dy = \int (0.611\, sen(6.62t - 3.1) + 0.0002)dt,$$
$$y(t) = -0.0922\, cos(6.62t - 3.1) + 0.0002t + K.$$

Thus, the equation that models the position of the mass as a function of time is:

$$y(t) = -0.092\, cos(6.62t - 3.1) + 0.0002t + 0.11 \tag{14}$$

Finally, we proceed to find the numerical differentiation by applying centered divided differences with an error of order $o(\square^4)$ at five steps, obtaining the second derivative for each time point using the expression given in (3).

Although the instrument records data every $0.02\ s$, we adjust the data to time intervals of *0.08* seconds to improve the approximation of the second numerical derivative.

The data are placed in a GeoGebra spreadsheet, adjusting the times to larger steps (h = 0.08), graphed, and a sinusoidal regression model is obtained to represent the second-order differential equation in the following form:

$$y'' = f(t) = Asen(\omega t + \varphi) + D \tag{15}$$

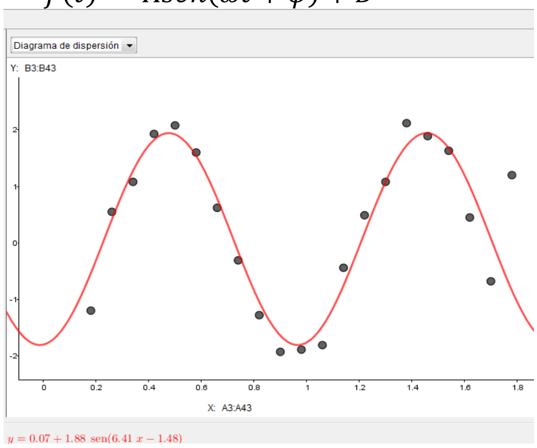

Fig. 8 Graph in GeoGebra of the second numerical differentiation and its corresponding sinusoidal regression representing the second-order differential equation

The coefficients of determination R^2 are determined for each type of regression or second-order differential equation.

Análisis de datos

Estadísticas

MediaX	0.98
MediaY	0.3419
Sx	0.4964
Sy	1.3818
r	0.0775
ρ	0.0494
nVarX	4.928
nVarY	38.1857
nCov	1.0632
R²	0.8362
SECs	6.2532

Fig. 9 Data analysis verifying the coefficient of determination $R^2 = 0.99$ for the second numerical differentiation

Applying double integration to (15) gives rise to two integration constants that are determined using the initial conditions of position $y(0) = 0.2$ and velocity $y'(0) = 0$.

$$\frac{d^2y}{dt^2} = 1.88\,\mathrm{sen}(6.41t - 1.5) + 0.07$$

Integrating the equation, we have:

$$\int \frac{d^2y}{dt^2}dt = \int (1.88\,\mathrm{sen}(6.41t - 1.5) + 0.07)\,dt,$$

$$\frac{dy}{dt} = -\frac{1.88}{6.41}\cos(6.41t - 1.5) + 0.07t + K_1.$$

Applying the initial condition:

$$\frac{dy}{dt}(0) = -\frac{1.88}{6.41}\cos(6.41(0) - 1.5) + 0.07(0) + K_1 = 0,$$

we obtain $K_1 = 0.027$.

Thus, we have:

$$\frac{dy}{dt} = -\frac{1.88}{6.41}\cos(6.41t - 1.5) + 0.07t + 0.027.$$

From this, we have that:

$$\int \frac{dy}{dt}dt = \int \left(-\frac{1.88}{6.41}cos(6.41t - 1.5) + 0.07t + 0.027\right)dt,$$

$$y(t) = -\frac{1.88}{41.0881}sen(6.41t - 1.5) + 0.035t^2 + 0.027t + K_2.$$

Applying the initial condition:

$$\frac{dy}{dt}(0) = -\frac{1.88}{6.41}cos(6.41(0) - 1.5) + 0.07(0) + K_1 = 0,$$

we obtain $K_1 = 0.027$.

Thus, we have:

$$\frac{dy}{dt} = -\frac{1.88}{6.41}cos(6.41t - 1.5) + 0.07t + 0.027.$$

From this, we have that:

$$\int \frac{dy}{dt}dt = \int \left(-\frac{1.88}{6.41}cos(6.41t - 1.5) + 0.07t + 0.027\right)dt,$$

$$y(t) = -\frac{1.88}{41.0881}sen(6.41t - 1.5) + 0.035t^2 + 0.027t + K_2.$$

Applying the initial condition of the position:

$$x(0) = -\frac{1.88}{41.0881}sen(6.41(0) - 1.5) + 0.035(0)^2 + 0.027(0) + K_2 = 0.2,$$

we obtain $\mathrm{K}_2 = 0.154$.

Thus, we have:

$$y(t) = -0.046\,sin(6.41t - 1.5) + 0.035t^2 + 0.027t + 0.154 \quad (16)$$

5. Results

The analysis uses the results obtained from the mathematical expressions (11), (12), (14), and (16), followed by a comparative analysis as follows.

Table 1 Mathematical models obtained during each phase of the process

Models	Differential Equation and its Solution
Model 1: Theoretical Model	$0.2y'' + 7.26y = 0$ $y(t) = 0.2\cos(6.025\,t)$
Model 2: Simulation-Regression (Using PhET simulator)	No differential equation generated $y(t) = 0.19\,sen(7.76t + 1.56) + 0.01$
Models	Differential Equation and its Solution
Model 1: Theoretical Model	$0.2y'' + 7.26y = 0$ $y(t) = 0.2\cos(6.025\,t)$
Model 2: Simulation-Regression (Using PhET simulator)	No differential equation generated. $y(t) = 0.19\,sen(7.76t + 1.56) + 0.01$
Model 3: Numerical Differentiation - 1st Order	$y' = 0.611\sin(6.62t - 3.1) + 0.0002$ $y(t) = -0.092\cos(6.62t - 3.1) + 0.0002t + 0.108$
Model 4: Numerical Differentiation - 2nd Order Model	$y'' = 1.88\,\mathrm{sen}(6.41t - 1.5) + 0.07$ $y(t) = -0.046\,\mathrm{sen}(6.41t - 1.5) + 0.035t^2 + 0.027t + 0.154$

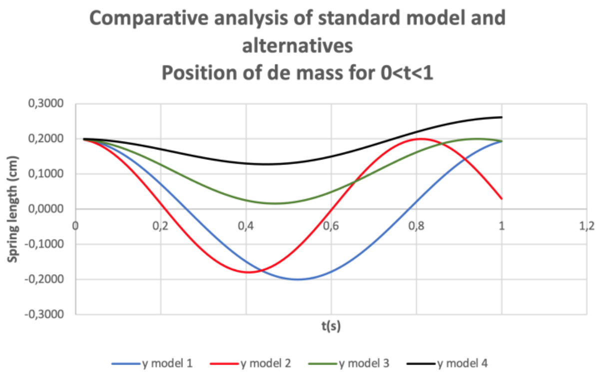

Fig. 10 Comparative analysis of the strategies for the vertical position of the mass with respect to time, implementing the four options

Finally, we perform a comparative analysis using the common questions posed in the four models. In addition, model 1 serves as the reference model to compare how far the data was obtained from models 2, 3, and 4.

Table 2 Coefficients of determination of models 2, 3, and 4 compared with theoretical model 1

Models compared to M1	M1 Model 2: PhET	Model 3: 1st Order	Model 4: 2nd Order
R^2	0.4392	0.8868	0.6163

When compared with the data recorded by the proximity sensor, we have:

Table 3 Coefficients of determination of models 2, 3, and 4 compared with experimental data

Models compared to recorded dat:	Model 1: Theoretical Model	Model 2: PhET	Model 3: DN order 1	Model 4: 2nd Order
R^2	0.9211	0.6870	0.9962	0.8155

The analysis of Tables 1 and 3 reveals a better approximation with model 3 using 1st-order numerical differentiation when comparing Models 2, 3, and 4 with the theoretical or reference model 1, with a coefficient of determination of $R^2 = 0.8868$. When compared to the actual data obtained with the proximity sensor, the 1st-order numerical differentiation model 3 achieves a very good approximation with a coefficient of determination of $R^2 = 0.9962$, but the 2nd-order numerical differentiation model 4 (the novelty of this research) achieves a coefficient of determination of $R^2 = 0.8155$, fulfilling the main objective of this project.

6. Conclusion

The developed didactic strategy for the teaching-learning process of differential equations through mathematical modeling incorporated activities that involved transitions between the pseudo-concrete, physical, and mathematical domains, as described in the mathematical modeling cycle proposed by [4], as the main components of the process, complemented with the procedures in each worked model and evaluation.

Through the evaluation and control carried out throughout the execution of the strategy, and considering the results of the comparative analysis between models 2, 3, and 4 concerning the reference model 1, as well as the comparison of all four models with the original data record, the fulfillment of the research objective and problem-solving is positively valued.

Note that this research did not address all the problems related to the teaching-learning process of differential equations. Furthermore, it did not cover aspects related to damping and external forces addressed in the theoretical models of classical differential equations textbooks, which opens up possibilities for future research in these areas and leaves open issues that need further investigation.

During the laboratory practice for models 3 and 4, damping forces that could affect the motion under normal conditions were not considered, which could influence the system proportionally to $x'(t)$. Therefore, future analyses could involve an additional term in the differential equation described in (1), such as $\beta x'$, to account for influences from the laboratory environment (external vibrations, temperature, acoustics, etc.).

Improvements can be made to the estimation obtained using the second-order finite difference formulas described in (3). One approach is to increase the step size h because small intervals with possible outliers or inconsistent data can lead to difficulties in calculating the second numerical derivative. Another approach is to restrict the time domain to avoid issues in the calculations and ensure that the second numerical derivative is meaningful.

Analogously, the experiment could be conducted with a pendulum system using the theoretical model, simulation, and first- and second-order numerical differentiation.

Note that the system studied in this research assumes an indefinite mass motion, but this is not the case in reality. However, there is a damping effect, as observed in equation (1), where the parameter γ is nonzero. This leads to the auxiliary equation having three distinct scenarios, determined by analyzing its discriminant:

1. Overdamped vibrations;
2. Critically damped vibrations;
3. Underdamped vibrations.

These scenarios can be mathematically modeled, provided that the data collection infrastructure is available. One practical application of these studies is in motorcycle suspension systems, as depicted in Fig. 11.

Fig. 11 Motorcycle suspension system [12]

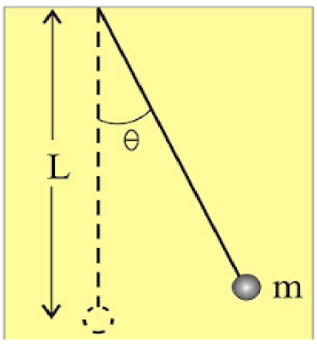

Fig. 12 Simple pendulum system (http://rsefalicante.umh.es/LaboratorioPendulo/Pendulo1.htm)

This research, conducted with a mass-spring system, could also be carried out with a simple pendulum system, where a mass m (with negligible string mass) is suspended from a string. The equations of motion allow for mathematical modeling of the system with small displacements (angle of rotation θ in radians as a function of time, θ = f(t), resulting in a second-order differential equation. Fig. 12 shows this.

Additionally, the solution of an RLC circuit can be applied, as shown in Fig. 13, where R represents resistance, L – inductance, and C – the capacitance of the circuit. The mathematical expression governing the behavior of the electric charge q as a function of time, q = f(t), would involve the effects of damping, as in the previous case observed in Fig. 13.

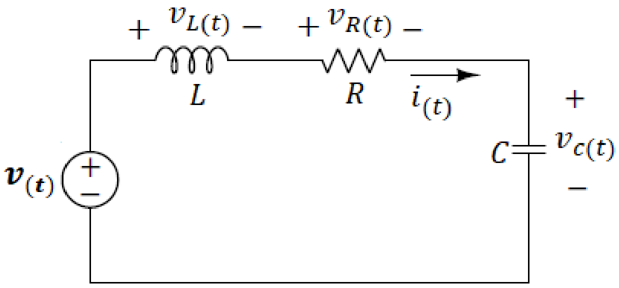

Fig. 13 RLC circuit [13]

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