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<https://doi.org/10.55463/issn.1674-2974.50.2.3>

A Comparison of the Adomian Decomposition Method and Variational Iteration Method for a Two-Dimensional Nonlinear Wave Equation

Sanaullah Dehraj¹, Asgher Ali Maitlo¹, Wajid Ahmed Siyal², Muhammad Memon¹, Lubna Naz Arain³, Laraib Arain⁴, Kamran Umrani¹

¹ Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology, 67480 Nawabshah, Sindh, Pakistan

² IBA University, Sukkur (IBA Community College), Sindh, Pakistan

³ Department of Natural Sciences, Begum Nusrat Bhutto Women Univeristy, Sukkur, Sindh, Pakistan

⁴ General Faculty, Shaheed Benazir Bhutto University, Shaheed Benazirabad, Sindh, Pakistan

Received: December 16, 2022 ▪ Review: January 10, 2023 ▪ Accepted: February 4, 2023 ▪ Published: February 27, 2023

Abstract: This paper studies a two-dimensional wave equation in the presence of power and derivative nonlinearity subject to suitable initial conditions. It aims to construct the exact-analytical solution of a two-dimensional nonlinear wave equation using two different semi-analytical methods and to compare the obtained results. First, a well-known Adomian decomposition method (ADM) based on operators is employed. Many researchers use the ADM to investigate several scientific applications, and this method straightforwardly attacks the problem without using linearization, discretization, perturbation, or any other restrictive assumption that may change the physical behavior of the problem. Second, the variational iteration method (VIM) also provides rapid convergent successive approximations of the closed-form solutions if it exists; otherwise, it provides an approximation of a high degree accuracy level even in case of few obtained iterations. The obtained results are drawn graphically and presented in the tables. Both methods provide almost the same solutions in each nonlinearity case, and VIM has computational advantages over ADM in computation size. Further, ADM and VIM provide a series of solutions that converge in a very small time domain, which limits these methods.

Keywords: Adomian decomposition method, variational iteration method, approximations, partial differential equations, two-dimensional wave equation.

一類二維非線性波動方程的阿多米安分解法與變分迭代法的比較

摘要：本文研究了在合適的初始條件下，存在冪和導數非線性的二維波動方程。旨在利用兩種不同的半解析方法構造二維非線性波動方程的精確解析解，並比較所得結果。首先，採用著名的基於算子的阿多米安分解方法。許多研究人員使用阿多姆分解法來研究幾個科學應用，這種方法直接解決問題，而不使用線性化、離散化、擾動或任何其他可能改變問題物理行為的限制性假設。其次，變分迭代法還提供了封閉形式解的快速收斂逐次逼近（如果存在的話）；否則，即使在獲得的迭代很少的情況下，它也提供了高精度水平的近似值。獲得的結果以圖形方式繪製並顯示在表格中。兩種方法在每種非線性情況下都提供幾乎相同的解



決方案，並且變分迭代法在計算量方面比阿多姆分解法具有計算優勢。此外，阿多姆分解法和變分迭代法提供了一系列在非常小的時域內收斂的解決方案，這限制了這些方法。

关键词：阿多姆安分解法、变分迭代法、逼近法、偏微分方程、二维波动方程。

1. Introduction

Partial differential equations (PDEs) can govern most of the phenomena that arise in scientific, engineering, and mathematical physical fields. For example, PDEs describe the physical phenomena of fluid dynamics, quantum mechanics, electricity, plasma physics, propagation of shallow water waves, and heat flow. All of these phenomena are put in the form of linear or nonlinear equations. Many methods and tools have been proposed to find exact solutions to such PDEs, but most of the time, it is not easy to find the exact solution of PDEs due to many difficulties such as the presence of nonlinear terms, non-classical boundary data, inhomogeneous terms, and mixed order derivatives. For such differential equations, researchers propose exact-analytical or numerical solutions such as two timescale perturbation methods [1-5], homotopy perturbation method [6-10], Adomian decomposition method (ADM) [11-16], variational iteration method [17-20], reduced differential transform method [21-22], modified ADM [23-25], numerical methods [26-28], and many others. Recently many researchers used the ADM to construct an exact (if it exists) or exact-analytical solution for many physical models. The ADM can apply to all types of differential equations straightforwardly, and this method does not require any unnecessary assumptions that may change the geometrical and physical aspects of the problem. This method is applicable for homogenous and inhomogeneous boundary conditions, and there is no need to convert inhomogeneous boundary conditions to homogeneous conditions as required in the method of separation of variables. Despite many advantages of this method, there is the main issue of the region and the rate of convergence. Due to the slow convergence rate in a wider time domain, it provides an erroneous solution in that domain of time. Recently many authors [29-32] have used the Adomian decompositions method to study linear and nonlinear ordinary and partial differential equations, fractional differential equations, delay differential equations, stochastic differential equations, and integrodifferential equations. The variational iteration method is another powerful tool used frequently to study all types of mathematical models, and this method does not require any transformation of the problem and can be used straightforwardly like ADM. In the VIM, there is no separate treatment for nonlinear terms as required. Adomian polynomials are required in the ADM, which

increase the computational time. Recently many comparative studies [33-38] have been done to investigate the efficiency, reliability, and accuracy of different approximated methods for many linear and nonlinear mathematical models.

In this work, we will study a two-dimensional wave equation in the presence of power and derivative nonlinearity subject to suitable initial conditions. Let us consider a two-dimensional nonlinear wave equation with suitable initial conditions given as

$$u_{tt} = c^2(u_{xx} + u_{yy}) + F(u) \quad (1)$$

Subjects to initial conditions are considered as $u(x, y, 0) = f(x, y)$ and $u_t(x, y, 0) = g(x, y)$ (2) where $u(x, y, 0)$ and $u_t(x, y, 0)$ are the initial displacement and velocity, respectively; differentiable functions f and g are twice differentiable functions;

$c = \sqrt{\frac{T}{\rho}}$ is wave speed; T and ρ are constant tension and density per unit length, and finally $F(u)$ represents the nonlinear term in general form. This paper aims to construct the approximations of the solutions for a two-dimensional homogeneous nonlinear wave equation. The approximated solutions are obtained by ADM and VIM, and obtained results are compared in terms of tables and graphical illustrations.

2. Methodology

In this section, we will discuss the methods to be applied for constructing the approximations of the solution of the problem given in Eqs. (1) and (2).

2.1. Adomian Decomposition Method (ADM)

To apply the ADM to Eqs. (1) and (2), we first write them in terms of the operator as

$$L_t u(x, y, t) = c^2 [L_x(u(x, y, t)) + L_y(u(x, y, t))] + F(u) \quad (3)$$

where $L_t = u_{tt}$, $L_x = u_{xx}$ and $L_y = u_{yy}$; the inverse operator L_t^{-1} is the multiple integral that is $L_t^{-1} = \int_0^t \int_0^t dt dt$. By applying the inverse differential operator L_t^{-1} on both sides of Eq. (3) and using Eq. (2), it yields:

$$u(x, y, t) = f(x, y) + tg(x, y) + c^2 L_t^{-1} [L_x(u(x, y, t)) + L_y(u(x, y, t))] + L_t^{-1} [F(u)] \quad (4)$$

Let us suppose that the solution of Eq. (4) is given in series form:

$$u(x, y, t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \dots \quad (5)$$

By setting Eq. (5) into Eq. (4), it yields:

$$\begin{aligned} & \sum_{n=0}^{\infty} u_n(x, y, t) \\ &= f + tg + c^2 L_t^{-1} \left[\left(L_x \left(\sum_{n=0}^{\infty} u_n \right) + L_y \left(\sum_{n=0}^{\infty} u_n \right) \right) \right] \\ &+ L_t^{-1} [F(u)] \end{aligned} \quad (6)$$

On expanding the summation in Eq. (6) and by collecting the terms outside the operators, which are considered zeroth-order approximations and the remaining terms can be obtained using recursive relations:

$$\begin{aligned} u_0 &= (f + tg), \text{ and } u_{n+1} \\ &= c^2 L_t^{-1} \left[\left(L_x \left(\sum_{n=0}^{\infty} u_n \right) \right. \right. \\ &\quad \left. \left. + L_y \left(\sum_{n=0}^{\infty} u_n \right) \right) \right] + L_t^{-1} (A_n), n \\ &= 0, 1, 2, \dots \end{aligned} \quad (7)$$

where the Adomian polynomials for nonlinear terms $F(u)$ can be obtained as

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F \left(\sum_{i=0}^n \lambda^i u_i \right) \right]_{\lambda=0} \quad \text{for } n = 0, 1, 2, \dots \quad (8)$$

After computing all the components, the required approximated solution can be constructed using Eq. (5). For detail, an understanding of the ADM reader is referred to [39].

2.2. Variational Iteration Method (VIM)

Consider the differential equation in operators form given as

$$L(u) + N(u) = g \quad (9)$$

where L is the linear operator, N represents the nonlinear part of the differential equation and g is the source term of the differential equation. The correctional function for solving the nonlinear partial differential equation given in Eq. (9) is defined as

$$\begin{aligned} u_{n+1}(t) &= u_n(t) \\ &+ \int_0^t \lambda(\xi) [(Lu_n(\xi) + Nu_n(\xi) \\ &- g(\xi))] d\xi \end{aligned} \quad (10)$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variation theory [40]. After computing the components using Eq. (10), make the sum of all components; it will provide an approximate solution to a given problem.

3. Results – Approximations of the Solutions

In this section, we will construct the approximate

solutions for different cases of nonlinearity for a two-dimensional wave equation as given in Eq. (1), subject to the initial conditions given in Eq. (2) via ADM and VIM.

3.1. Problem 1: A Two-Dimensional Wave Equation with Power Nonlinearity

Let us consider a two-dimensional wave equation with power nonlinearity:

$$u_{tt} = \frac{1}{2} (u_{xx} + u_{yy}) + u^2 \quad (11)$$

Subjects to the specified initial conditions are considered as:

$$u(x, y, 0) = 1 \text{ and } u_t(x, y, 0) = e^{x+y} \quad (12)$$

3.1.1. Solution by the ADM

To determine the terms of the decomposition of the solution $u(x, y, t)$ of Eq. (5) we use Eq. (7), the first approximation is obtained by the initial conditions and is given as

$$u_0(x, y, t) = (1 + te^{x+y}) \quad (13)$$

Furthermore, the other remaining terms can be obtained using the recursive relation given as

$$\begin{aligned} u_{n+1}(x, y, t) &= \frac{1}{2} L_t^{-1} [L_x(u_n) + L_y] \\ &+ L_t^{-1} (A_n), \end{aligned} \quad (14)$$

The zeroth order Adomian polynomial for the nonlinear term can be obtained using Eq. (8) and is given as

$$A_0 = u_0^2 = (1 + 2te^{x+y} + t^2 e^{2(x+y)}) \quad (15)$$

Thus, to obtain the first approximation of the component we put it into Eq. (14) and by using $u_1(x, y, t)_{n=0}$ (13) and (15), it yields:

$$u_1(x, y, t) = \left(\frac{t^2}{2} + \frac{t^3}{2} e^{x+y} + \frac{t^4}{12} e^{2(x+y)} \right) \quad (16)$$

To compute the next component of the solution, first we obtain the Adomian polynomial by using Eq. (8); it yields:

$$A_1 = 2u_0 u_1 = \left(\frac{t^2 + 2t^3 e^{x+y}}{6} + \frac{7}{6} t^4 e^{2(x+y)} + \frac{1}{6} t^5 e^{3(x+y)} \right) \quad (17)$$

The next component of the solution can be obtained by setting $n = 1$ into Eq. (14) and by using the (16) and (17), it yields:

$$u_2(x, y, t) = \left(\frac{t^4}{12} + \frac{t^5}{8} e^{x+y} \right. \\ \left. + \frac{t^6}{20} e^{2(x+y)} + \frac{t^7}{252} e^{3(x+y)} \right) \quad (18)$$

The third component of the solutions can be obtained in a similar manner but it takes more computational work. Thus, the solution up to second approximations is given as

$$\begin{aligned} u(x, y, t) &= u_0 + u_1 + u_2 + \dots \\ &= \left(1 + \frac{t^2}{2} + \frac{t^4}{12} \right) + \left(t + \frac{t^3}{2} + \frac{t^5}{8} \right) e^{x+y} \\ &+ \left(\frac{t^4}{12} + \frac{t^6}{20} \right) e^{2(x+y)} + \frac{t^7}{252} e^{3(x+y)} \\ &+ \dots \end{aligned} \quad (19)$$

3.1.2. Solution by VIM

Next, we will construct the solution of the same initial value problem given in Eqs. (11) and (12) by using VIM. Thus by variational functional for Eq. (11) is given as follows:

$$u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^t \lambda(\xi) \left[\left(\frac{\partial^2 u_n}{\partial \xi^2} - \frac{1}{2} \frac{\partial^2 u_n}{\partial x^2} - \frac{1}{2} \frac{\partial^2 u_n}{\partial y^2} - u_n^2 \right) \right] d\xi \tag{20}$$

The stationary conditions for the problem given in Eq. (20) are given as follows:

$$(1 - \lambda')|_{\xi=t} = 0, \lambda|_{\xi=t} = 0 \text{ and } \lambda''|_{\xi=t} = 0 \tag{21}$$

Thus, the value of Lagrange's multiplier is given below:

$$\lambda = (\xi - t) \tag{22}$$

To compute the first iteration we use Eq. (13) into (20) and the Lagrange value given in Eq. (22); it yields:

$$u_1(x, y, t) = \left(1 + \frac{t^2}{2} + te^{x+y} + \frac{t^3}{3} e^{x+y} + \frac{t^4}{12} e^{2(x+y)} \right) \tag{23}$$

Similarly for $n = 1$ Eq. (20), the next iteration is

$$u_2 = \left(1 + \frac{t^2}{2} + \frac{t^4}{12} + \frac{t^6}{120} + \left(t + \frac{t^3}{2} + \frac{t^5}{10} + \frac{t^7}{126} + \frac{t^9}{90} \right) e^{x+y} + \left(\frac{t^4}{12} + \frac{t^6}{36} + \frac{t^8}{288} \right) e^{2(x+y)} + \left(\frac{t^7}{252} + \frac{t^9}{1296} \right) \left(\frac{t^4}{12} + \frac{t^6}{36} + \frac{t^8}{288} \right) e^{3(x+y)} + \frac{t^{10}}{12960} e^{4(x+y)} \right) \tag{24}$$

The same process can be continued for computing higher components, but it requires more computational work. Thus the approximate solution up to second order is given as

$$u = \left(1 + \frac{t^2}{2} + \frac{t^4}{12} + \frac{t^6}{120} + \left(t + \frac{t^3}{2} + \frac{t^5}{10} + \frac{t^7}{126} + \frac{t^9}{90} \right) e^{x+y} + \left(\frac{t^4}{12} + \frac{t^6}{36} + \frac{t^8}{288} \right) e^{2(x+y)} + \left(\frac{t^7}{252} + \frac{t^9}{1296} \right) \left(\frac{t^4}{12} + \frac{t^6}{36} + \frac{t^8}{288} \right) e^{3(x+y)} + \frac{t^{10}}{12960} e^{4(x+y)} + \dots \right) \tag{25}$$

It is essential to mention that approximated solutions for a two-dimensional wave equation

obtained by ADM and VIM, as given in Eq. (19) and (25), satisfy both initial conditions of the problem.

It can be easily observed from Table 1 that the approximations of solutions for a two-dimensional wave equation with power nonlinearity obtained by ADM and VIM are correct up to 6 decimal places.

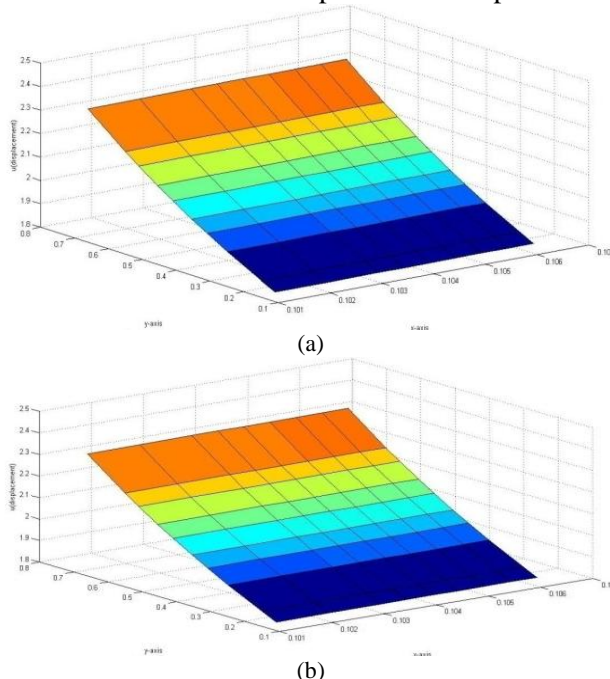


Fig. 1 Approximations of the solutions for the two-dimensional wave equation with power nonlinearity: (a) ADM and (b) VIM for $x = [0.101, 0.1095], y = [0.151, 0.66]$ and $t = 0.05$

The graphical illustration in Fig. 1 of the approximate solutions obtained by ADM and VIM clearly shows that both methods yield almost similar approximate solutions, which are correct up to 6 decimal places, as shown in Table 1.

Table 1 Approximations of the solutions for power nonlinearity case

t	x	y	ADM	VIM	ADM-VIM
0.01	0.101	0.151	1.0129166058	1.0129166058	0
0.02	0.102	0.11	1.0249032209	1.0249032209	0
0.03	0.103	0.25	1.0431266598	1.0431266589	0.0000000008
0.04	0.104	0.31	1.0613028845	1.0613028805	0.0000000039
0.05	0.105	0.355	1.0804357265	1.0804357137	0.0000000128
0.06	0.106	0.41	1.1023027824	1.1023027483	0.0000000341
0.07	0.107	0.455	1.1252444198	1.1252443415	0.0000000782
0.08	0.108	0.51	1.1516619341	1.1577982522	0.0000001715
0.09	0.109	0.55	1.1781288179	1.1781285056	0.0000003122
0.1	0.1095	0.66	1.2212398360	1.2212392278	0.0000006082

3.2. Problem 2: A Two-Dimensional Wave Equation with Derivative Nonlinearity

Let us consider a two-dimensional wave equation with power nonlinearity as follows:

$$u_{tt} = \frac{1}{2}(u_{xx} + u_{yy}) + u_x^2 \tag{26}$$

Subjects to the specified initial conditions are considered as

$$u(x, y, 0) = 1 \text{ and } u_t(x, y, 0) = e^{x+y} \quad (27)$$

3.2.1. Solution by the ADM

To determine the first term of the decomposition of the solution $u(x, y, t)$ of Eq. (5) we use Eq. (7); it yields:

$$u_0(x, y, t) = (1 + te^{x+y}) \quad (28)$$

Other remaining terms can be obtained using the recursive relation given as

$$u_{n+1}(x, y, t) = \frac{1}{2} L_t^{-1} \left[\begin{array}{l} L_x(u_n) \\ + L_y(u_n) \end{array} \right] + L_t^{-1}(A_n), \text{ for } n \geq 0 \quad (29)$$

The zeroth order Adomian polynomial for the nonlinear term u_x^2 can be obtained using Eq. (8) and is given as follows:

$$A_0 = (t^2 e^{2(x+y)}) \quad (30)$$

The first component of the solution can be obtained by setting $n = 0$ into Eq. (29) and by using Eq. (28) and (30), it yields:

$$u_1(x, y, t) = \left(\frac{t^3}{6} e^{x+y} + \frac{t^4}{12} e^{2(x+y)} \right) \quad (31)$$

For the next component first, we compute the Adomian polynomial by using Eq. (8); thus, it yields:

$$B_1 = \left(\frac{1}{4} t^4 e^{2(x+y)} + \frac{1}{3} t^5 e^{3(x+y)} \right) \quad (32)$$

Now, by setting $n = 1$ into Eq. (29) and then by using Eq. (31) and (32), it yields:

$$u_2(x, y, t) = \left(\frac{t^5}{120} e^{x+y} + \frac{t^6}{45} e^{2(x+y)} + \frac{t^7}{126} e^{3(x+y)} \right) \quad (33)$$

In a similar manner, the third component is obtained as

$$u_3(x, y, t) = \left(\frac{t^7}{5040} e^{x+y} + \frac{t^8}{420} e^{2(x+y)} + \frac{227t^9}{90720} e^{3(x+y)} + \frac{19t^{10}}{22680} e^{4(x+y)} \right) \quad (34)$$

Thus, the approximate solution for a two-dimensional wave equation with derivative nonlinearity is given as:

$$\begin{aligned} (x, y, t) &= u_0 + u_1 + u_2 + \dots \\ &= 1 + \left(t + \frac{t^3}{6} + \frac{t^5}{120} + \frac{t^7}{5040} \right) e^{x+y} \\ &\quad + \left(\frac{t^4}{12} + \frac{t^6}{45} + \frac{t^8}{420} \right) e^{2(x+y)} \\ &\quad + \left(\frac{t^7}{126} + \frac{227t^9}{90720} \right) e^{3(x+y)} \\ &\quad + \frac{19t^{10}}{22680} e^{4(x+y)} + \dots \end{aligned} \quad (35)$$

3.2.2. Solution by VIM

Next, we will construct the approximate solution problem given in Eqs. (26) and (27) by using VIM.

Thus, by variational functional for Eq. (26) and the Lagrange value as given in Eq. (22) is given as

$$u_{n+1}(x, y, t) = u_n + \int_0^t (\xi - t) \left[\left(\frac{\partial^2 u_n}{\partial \xi^2} - \frac{1}{2} \frac{\partial^2 u_n}{\partial x^2} \right) - \left(-\frac{1}{2} \frac{\partial^2 u_n}{\partial y^2} - \left(\frac{\partial u_n}{\partial x} \right)^2 \right) \right] d\xi \quad (36)$$

For the first approximation, put $n = 0$ into Eq. (36) and use Eq. (28); it yields:

$$u_1(x, y, t) = \left(1 + te^{x+y} + \frac{t^3}{6} e^{x+y} + \frac{t^4}{12} e^{2(x+y)} \right) \quad (37)$$

By setting $n = 1$ into Eq. (36) and use Eq. (37); it yields:

$$\begin{aligned} u_2(x, y, t) &= 1 + \left(t + \frac{t^3}{6} + \frac{t^5}{120} \right) e^{x+y} \\ &\quad + \left(\frac{t^4}{12} + \frac{t^6}{45} + \frac{13t^8}{19152} \right) e^{2(x+y)} \\ &\quad + \left(\frac{t^7}{126} + \frac{t^9}{1296} \right) e^{3(x+y)} \\ &\quad + \frac{t^{10}}{3240} e^{4(x+y)} \end{aligned} \quad (38)$$

Thus, the final approximate solution is given as

$$\begin{aligned} u(x, y, t) &= u_0 + u_1 + u_2 + \dots \\ &= 1 + \left(t + \frac{t^3}{6} + \frac{t^5}{120} \right) e^{x+y} \\ &\quad + \left(\frac{t^4}{12} + \frac{t^6}{45} + \frac{13t^8}{19152} \right) e^{2(x+y)} \\ &\quad + \left(\frac{t^7}{126} + \frac{t^9}{1296} \right) e^{3(x+y)} \\ &\quad + \frac{t^{10}}{3240} e^{4(x+y)} + \dots \end{aligned} \quad (39)$$

The graphical illustration in Fig. 2 of the approximate solutions obtained for a two-dimensional wave equation with derivative nonlinearity by ADM and VIM yields similar approximations.

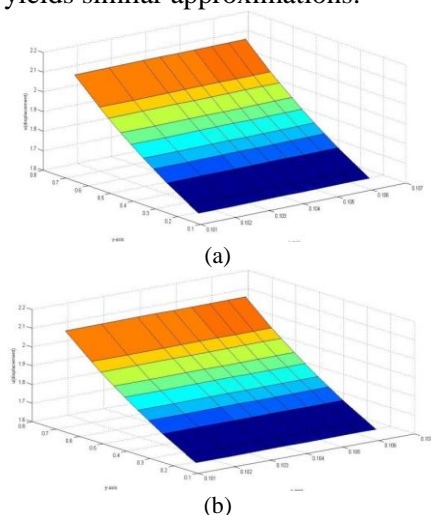


Fig. 2 Approximations of the solutions for two-dimensional wave equations with derivative nonlinearity: (a) ADM and (b) VIM for $x = [0.101, 0.106]$, $y = [0.151, 0.66]$ and $t = 0.05$

Table 2: Approximations of solutions for derivative nonlinearity case

t	x	y	ADM	VIM	ADM-VIM
0.01	0.101	0.151	1.0128661761	1.0128661761	0
0.02	0.102	0.11	1.0246999140	1.0246999140	0
0.03	0.103	0.25	1.0426637910	1.0426637910	0
0.04	0.104	0.31	1.0604704108	1.0604704108	0
0.05	0.105	0.355	1.0791192428	1.0791192428	0
0.06	0.106	0.41	1.1003811657	1.1003811657	0
0.07	0.107	0.455	1.1225920058	1.1225920058	0
0.08	0.108	0.51	1.1481421146	1.1481421146	0
0.09	0.109	0.55	1.1736039265	1.1736039265	0
0.1	0.1095	0.66	1.2155118204	1.2155118203	0.00000000001

It can easily be observed from Table 2 that the approximations of the solutions obtained by ADM and VIM are correct up to 10 decimal places, and both methods effectively solve a two-dimensional nonlinear wave equation.

3.3. Problem 3: A Two-Dimensional Wave Equation by Derivative Multiple Nonlinearity

Let us consider a two-dimensional wave equation with derivative multiple nonlinearities as follows:

$$u_{tt} = \frac{1}{2}(u_{xx} + u_{yy}) + uu_x \tag{40}$$

Subjects to the specified initial conditions are considered as

$$u(x, y, 0) = 1 \text{ and } u_t(x, y, 0) = e^{x+y} \tag{41}$$

3.3.1. Solution by the ADM

To determine the terms of the decomposition of the solution $u(x, y, t)$ of Eq. (5), we use Eq. (7); it yields:

$$u_0(x, y, t) = (1 + te^{x+y}) \tag{42}$$

Other remaining terms can be obtained using the recursive relation given as

$$u_{n+1}(x, y, t) = \frac{1}{2}L_t^{-1}[L_x(u_n) + L_y] + L_t^{-1}(A_n), \text{ for } n \geq 0 \tag{43}$$

The zeroth order Adomian polynomial for the nonlinear term uu_x can be obtained using Eq. (8) and is given as

$$A_0 = (te^{x+y} + t^2e^{2(x+y)}) \tag{44}$$

By setting $n=0$ into Eq. (43) and then by using Eq. (42) and (44), it yields

$$u_1(x, y, t) = \left(\frac{t^3}{3}e^{x+y} + \frac{t^4}{12}e^{2(x+y)}\right) \tag{45}$$

In a similar manner, the next two components are obtained as

$$u_2(x, y, t) = \left(\frac{t^5}{30}e^{x+y} + \frac{7t^6}{180}e^{2(x+y)} + \frac{t^7}{168}e^{3(x+y)}\right), \tag{46}$$

and the third component is as follows:

$$u_3(x, y, t) = \left(\frac{t^7}{630}e^{x+y} + \frac{17t^8}{2520}e^{2(x+y)} + \frac{37t^9}{6048}e^{3(x+y)} + \frac{11t^{10}}{2520}e^{4(x+y)}\right) \tag{47}$$

Thus, the approximated solution is given as

$$u = u_0 + u_1 + u_2 + \dots = 1 + \left(t + \frac{t^3}{3} + \frac{t^5}{30} + \frac{t^7}{630}\right)e^{x+y} + \left(\frac{t^4}{12} + \frac{7t^6}{180} + \frac{19t^8}{5040}\right)e^{2(x+y)} + \left(\frac{t^7}{168} + \frac{43t^9}{15120}\right)e^{3(x+y)} + \frac{19t^{10}}{45360}e^{4(x+y)} + \dots \tag{48}$$

3.3.2. Solution by VIM

Next, we will construct the approximate solution to the problem in Eqs. (40) and (41) by using VIM. Thus, by variational functional for Eq. (40) and the Lagrange value given in Eq. (22), it yields:

$$u_{n+1} = u_n + \int_0^t (\xi - t) \left[\begin{array}{c} \frac{\partial^2 u_n}{\partial \xi^2} \\ -\frac{1}{2} \frac{\partial^2 u_n}{\partial x^2} \\ -\frac{1}{2} \frac{\partial^2 u_n}{\partial y^2} - u_n \frac{\partial u_n}{\partial x} \end{array} \right] d\xi \tag{49}$$

For the first approximation, put $n=0$ into Eq. (49), and by using Eq. (42), it yields:

$$u_1(x, y, t) = \left(1 + te^{x+y} + \frac{t^3}{3}e^{x+y} + \frac{t^4}{12}e^{2(x+y)}\right) \tag{50}$$

Similarly, the following approximations are obtained by setting $n = 1$ into Eq. (49) and by using Eq. (50), it yields:

$$u_2(x, y, t) = 1 + \left(t + \frac{t^3}{3} + \frac{t^5}{30}\right)e^{x+y} + \left(\frac{t^4}{12} + \frac{7t^6}{180} + \frac{t^8}{504}\right)e^{2(x+y)} + \left(\frac{t^7}{168} + \frac{t^9}{864}\right)e^{3(x+y)} + \frac{t^{10}}{6480}e^{4(x+y)} \tag{51}$$

Thus, the final solution is given as

$$u = u_0 + u_1 + u_2 + \dots = 1 + \left(t + \frac{t^3}{3} + \frac{t^5}{30}\right)e^{x+y} + \left(\frac{t^4}{12} + \frac{7t^6}{180} + \frac{t^8}{504}\right)e^{2(x+y)} + \left(\frac{t^7}{168} + \frac{t^9}{864}\right)e^{3(x+y)} + \frac{t^{10}}{6480}e^{4(x+y)} + \dots \tag{52}$$

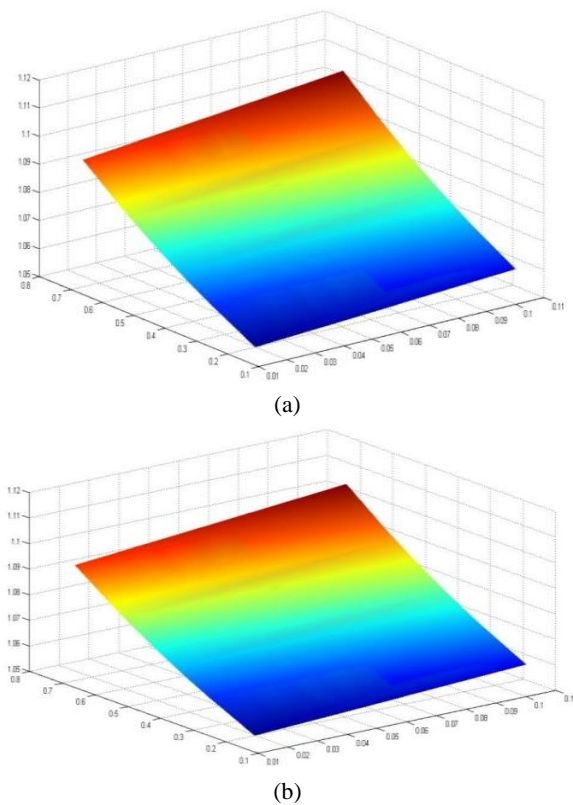


Fig. 3 Approximations of the solutions for two-dimensional wave equations with derivative multiple nonlinearities: (a) ADM and (b) VIM for $x = [0.101, 0.106]$, $y = [0.151, 0.66]$ and $t = 0.05$

The graphical illustration given in Fig. 3 of the approximate solutions obtained by ADM and VIM has a high accuracy level, demonstrating the effectiveness and reliability of both methods for nonlinear problems.

Table 3: Approximations of the solution for derivative multiple nonlinearity case

t	x	y	ADM	VIM	$ ADM-VIM $
0.01	0.101	0.151	1.0128663906	1.0128663906	0
0.02	0.102	0.11	1.0225529607	1.0225529607	0
0.03	0.103	0.25	1.0393108401	1.0393108401	0
0.04	0.104	0.31	1.0562283020	1.0562283020	0
0.05	0.105	0.355	1.0739117334	1.0739117334	0
0.06	0.106	0.41	1.0942143511	1.0942143511	0
0.07	0.107	0.455	1.1156045365	1.1156045365	0
0.08	0.108	0.51	1.1403632659	1.1403632659	0
0.09	0.109	0.55	1.1652764597	1.1652764597	0
0.1	0.1095	0.66	1.2061641587	1.2061641587	0

It can be easily observed from Table 3 that the approximations of the solutions obtained by ADM and VIM are correct up to 10 decimal places, and both methods effectively solve a two-dimensional nonlinear wave equation.

4. Conclusion

This paper examined a two-dimensional wave

equation in the presence of power and derivative nonlinearity with suitable initial conditions. The approximate solutions of a two-dimensional wave equation with power and derivative nonlinearity were obtained by the Adomian decomposition method and the variational iteration method. The main finding is that for the power nonlinearity case, the approximate solutions obtained by both methods yield correct solutions up to 6 decimal places, as given in Table 1. For the derivative nonlinearity case, the solutions obtained by both methods were similar up to 10 decimal places, as in Table 2. In the case of the derivative multiple nonlinearities, both methods provided solutions correct to 10 decimal places, as in Table 3. Both methods provide almost similar series solutions in each nonlinearity case that converge in a very small time domain, which is the main disadvantage of these methods. In the future, it is possible to compare these methods for the nonlinear fractional wave equation.

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