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Solution Analysis for D-Dimensional Cosmic String Coupled by Hyperbolic Scarf Plus Poschl-Teller and Manning-Rosen Potentials

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Abstract: Wave function and energy spectra for a d-dimension time-independent cosmic string with the radial hyperbolic scarf and angular Poschl-Teller non-central potential had been investigated with the hypergeometric method. The aim of this work was to obtain the solution and analyze the energy level, wave function, and its application in the Renyi entropy and the Schwarzschild black hole. The equation for d-dimension time-independent cosmic string was reduced into a d-dimension Schrodinger-like equation. The variable separation method was applied to resolve the equation into d's one-dimensional Schrodinger-like equations. Each of d's one-dimensional Schrodinger-like equations was analyzed by using the hypergeometric method to obtain wave function and energy equations. The energy equation was obtained from the solution of the radial part. Wave functions were obtained from the solution of the radial and angular parts. The energy levels were higher in the presence of Hyperbolic Scarf plus Poschl-Teller and Manning-Rosen potentials. Numerical results of energy levels showed that the energy level increased a similar amount to the increase of potential parameters. The form of the plot of energy levels as a function of the cosmic string parameter showed the quartic function. This solution was applied to determine the thermodynamic properties of the Schwarzschild black hole which has not been reported in previous works. This information can be used for future research of the properties of the Schwarzschild black hole.

Keywords: wave function, energy spectra, cosmic string, hyperbolic scarf potential.

双曲围巾加波斯菊板和曼宁-罗森势耦合的 D 维宇宙弦的解分析

摘要:
用超几何方法研究了具有径向双曲围巾和角波斯菊板非中心势的d维时间无关宇宙弦的波函数和能谱。这项工作的目的是获得解并分析能级、波函数及其在仁义熵和施瓦茨柴尔德黑洞中的应用。d维时间无关宇宙弦方程简化为d维薛定谔方程。应用变量分离法将方程分解为d的一维类薛定谔方程。使用超几何方法对d的每个一维类薛定谔方程进行分析，得到波函数和能量方程。能量方程是从径向部分的解中获得的。波函数是从径向和角部分的解中获得的。在双曲围巾加上波斯菊板和曼宁-
罗森电位存在的情况下，能量水平更高。能级的数值结果表明，能级的增加量与势参数的增加量相似。作为宇宙弦参数的函数的能级图的形式显示了四次函数。该解决方案用于确定史瓦西黑洞的热力学性质，这在以前的工作中尚未报道。这些信息可用于未来对史瓦西黑洞特性的研究。

关键词：波函数、能谱、宇宙弦、双曲围巾势。

1. Introduction

Cosmology and string theory are two fields of physics that together prove the possibility of microscopic explanations for the early history of the universe. Cosmological theory is based on the big bang phenomenon in the universe. The cosmic string is analogous to an object formed in the early universe. It is a topological defect that appeared in the phase

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transitions of the early universe [1, 2]. As a topological defect, the cosmic string has been studied in relativistic and non-relativistic systems, and the solutions using these systems have been solved by spherical and cylindrical coordinates [3-6]. The time-independent cosmic string equation is reduced to the Schrodinger-like equation in the case of the non-relativistic system, whereas it is reduced to the Dirac equation or the Klein Gordon equation in the relativistic system [7-9].

The central potential or non-central potential can be coupled with the time-independent cosmic string. The central potential is the potential energy that only consists of the radial part's potential, while the non-central potential consists of both the radial and angular part's potentials. Potentials such as Woods-Saxon [10], double ring-shaped oscillator [11], double ring-shaped Coulomb oscillator [12], Poschl-Teller double ring-shaped Coulomb potential, Coulomb-like scalar [13], harmonic oscillator plus Posch-Teller potential [14], and trigonometric non-central potentials [15] have been investigated in this case.

In this research, we investigated radial hyperbolic Scarf plus angular Poschl-Teller and Manning-Rosen non-central potentials that were coupled with the time-independent cosmic string equation. This gives a separable non-central potential, as shown in the equation below:

$$V(\mathrm{r},\theta_1,\theta_2,\theta_3,\theta_4)=\begin{pmatrix} V(\mathrm{r})+\dfrac{V(\theta_1)}{r^2\sin^2\theta_2\sin^2\theta_3\sin^2\theta_4} \\ +\dfrac{V(\theta_2)}{r^2\sin^2\theta_3\sin^2\theta_4}+\dfrac{V(\theta_3)}{r^2\sin^2\theta_4}+\dfrac{V(\theta_4)}{r^2}\end{pmatrix} \quad (1)$$

$V(\mathrm{r})$ is the modified pseudo harmonic.

$$V(\mathrm{r})=D_e\left[\frac{r}{r_e}-\frac{r_e}{r}+c\right]^2. \quad (2)$$

$$V(\theta_1)=\frac{\hbar^2}{2M}\left[\frac{\kappa_1(\kappa_1-1)}{\sin^2\theta_1}+\frac{\lambda_1(\lambda_1-1)}{\cos^2\theta_1}\right]. \quad (3a)$$

$$V(\theta_2)=\frac{\hbar^2}{2M}\left[\frac{\nu_2(\nu_2+1)}{\sin^2\theta_2}-2\mu_2\cot\theta_2\right]. \quad (3b)$$

$$V(\theta_1)=\frac{\hbar^2}{2M}\left[\frac{\nu_1(\nu_1+1)}{\cos^2\theta_1}-2\mu_1\tan\theta_1\right]. \quad (3c)$$

θ_i was changed to the ring-shaped potential.

The hyperbolic scarf potential includes the radial part's potential and the Poschl-Teller potential corresponds to the potential in the angular parts. The Poschl-Teller potential was used to describe the molecular vibration with $\kappa>1$ and $\lambda>1$, the hyperbolic Scarf potential with $b^2+a(a-1)>0$, and $2b\left(a-1/2\right)>0$. The Manning-Rosen potential was used to investigate the thermodynamic properties of the Schwarzschild black hole, with $\nu>1$ and $\mu>1$ [16-20]. The hyperbolic Scarf plus angular non-central potential is a separable potential. The time-independent cosmic string equation that is coupled with the 5-dimensional potential is solvable by using the variable separation method. The effect of the application of the radial hyperbolic Scarf plus angular Poschl-Teller and Manning-Rosen non-central potential in the equation of the d-dimensional time-independent cosmic string to the energy levels and wave functions was investigated. The energy equation and wave functions as a solution of the d-dimensional time-independent cosmic string can be obtained by using methods such as the Frobeneus method, factorization method, supersymmetry quantum mechanics method, and Nikiforov Uvarov method [2, 8, 21, 22]. In this research, we used the hypergeometric method to obtain energy and wave function equations.

The d-dimensional system explains that the unification of two fundamental forces, gravitational and electromagnetic, is involved in the creation of such a huge universe. More than 3+1 dimensions are needed for the description of gravity; the additional spatial dimension is from string theory [23]. The solution of the d-dimensional time-independent cosmic string equation will be investigated by reducing it to a d-dimensional Schrodinger-like equation. The solution of the d-dimensional equation in the relativistic system using the Klein Gordon or Dirac equation has been studied. Studies on the d-dimensional non-relativistic system using the Schrodinger equation have also been conducted [24-26].

In this research, the Renyi entropy of the system was also investigated. Renyi entropy was introduced by Renyi in 1960. Renyi entropy $\left(R_q\right)$ is a generalization of the Shannon entropy, which depends on a parameter, q. q is known as the Renyi index [27]. Renyi defined the entropy [28], where $\rho(r)=X^2(r)$ and $X(r)$ is ground state wave function.

In this paper, we investigated the solution of the d-dimensional time-independent cosmic string equation for the hyperbolic Scarf plus Poschl-Teller and Manning-Rosen non-central potentials using the hypergeometric method. Section 2 explains the basic theories about the d-dimensional time-independent cosmic string; the application of the d-dimensional time-independent cosmic string coupled with non-central potentials that are reduced into a d-dimensional Schrodinger-like equation with non-central potentials is solved by separation and hypergeometric methods. In Section 3, the results of the solution of the d-dimensional time-independent cosmic string for the hyperbolic Scarf plus Poschl-Teller and Manning-Rosen potentials are discussed, and in Section 4, we summarize the results.

2. Basic Theory

2.1. D-Dimensional Time-Independent Cosmic String Equation

The cosmic string equation will be investigated in the d-dimensional system. The metric of the cosmic string space-time for a 5-dimensional system that is obtained from a hyperspherical coordinate system is defined as:

$$\begin{aligned} x_1 &= r\cos\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \\ x_2 &= r\sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \\ x_3 &= r\cos\theta_2 \sin\theta_3 \sin\theta_4 \\ x_4 &= r\cos\theta_3 \sin\theta_4 \\ x_5 &= r\cos\theta_4 \end{aligned} \tag{4}$$

By taking $\theta_0 = r$ and by using the formula,

$$\begin{aligned} h &= \prod_{j=0}^{D-1} h_j \\ h_j^2 &= \sum_{i=1}^{D}\left(\frac{\partial x_i}{\partial \theta_j}\right)^2 \\ g_{jj} &= h_j^2 \\ ds^2 &= \sum_i^D (h_i dx_i)^2 \end{aligned} \tag{5}$$

where ds^2 is the square of infinitesimal element length.

For the 5-dimensional hyperspherical system with the coordinate axis given in equation (4), ds^2 obtained from equations (4) and (5) is given as:

$$ds^2 = \begin{pmatrix} dr^2 + r^2\alpha^2 \sin^2\theta_2 \sin^2\theta_3 \sin^2\theta_4 d\theta_1^2 \\ +r^2 \sin^2\theta_3 \sin^2\theta_4 d\theta_2^2 \\ +r^2 \sin^2\theta_4 d\theta_3^2 + r^2 d\theta_4^2 \end{pmatrix} \tag{6}$$

where $0 \le r \le \infty$, $0 \le \theta_i \le \pi$, $0 \le \theta_1 \le 2\pi$, and $i = 2,3,4$. The cosmic string parameter is given as: $\alpha = 1 - 4G\mu$, where μ is the linear mass density. The cosmic string parameter runs in the interval (0,1). By putting $\alpha = 1$ in the d-dimensional coordinates, the cosmic string space-time reduces to Minkowski space-time. To obtain the solution of the d-dimensional time-independent cosmic string equation, the variable separation method is applied to produce the time-dependent equation and space-coordinate dependent equation. In this research, we solve the spatial part from space-time and then obtain a time-independent cosmic string equation. The metric of the d-dimensional time-independent cosmic string is obtained by using the curvilinear system; $ds^2 = \sum_{ij} g_{ij} dq^i dq^j$ is obtained from equations (5) and (6) and is given as:

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta_4 & 0 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta_3\sin^2\theta_4 & 0 \\ 0 & 0 & 0 & 0 & r^2\alpha^2\sin^2\theta_2\sin^2\theta_3\sin^2\theta_4 \end{bmatrix} \tag{7}$$

The Laplacian of the 5-dimensional system in hyperspherical coordinates is obtained by using

$$\nabla_D^2 = \frac{1}{h}\sum_{j=0}^{D-1}\frac{\partial}{\partial\theta_j}\left(\frac{h}{h_j^2}\frac{\partial}{\partial\theta_j}\right) \tag{8}$$

By inserting the calculated parameters h and h_j^2 in equation (5) for the coordinate system in equation (4) into equation (8), we obtain Laplacian ∇^2 in 5-dimensional hyperspherical coordinates as:

$$\nabla_5^2 = \begin{bmatrix} \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\right) + \frac{1}{r^2\alpha^2\sin^2\theta_2\sin^2\theta_3\sin^2\theta_4}\frac{\partial}{\partial\theta_1}\left(\frac{\partial}{\partial\theta_1}\right) \\ +\frac{1}{r^4\sin\theta_2\sin^2\theta_3\sin^3\theta_4}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial}{\partial\theta_2}\right) \\ +\frac{1}{r^2\sin^2\theta_3\sin^2\theta_4}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial}{\partial\theta_3}\right) \\ +\frac{1}{r^2\sin^3\theta_4}\frac{\partial}{\partial\theta_4}\left(\sin^3\theta_4\frac{\partial}{\partial\theta_4}\right) \end{bmatrix} \tag{9}$$

The 5-dimensional Laplacian in hyperspherical coordinates is usable in a 5-dimensional time-independent cosmic string that is coupled with the non-central potential presented in equations (1-3).

2.2. D-Dimensional Time-Independent Cosmic String Coupled with Non-Central Potentials

The solution of the d-dimensional time-independent cosmic string for hyperbolic Scarf plus Poschl-Teller and Manning-Rosen non-central potentials will be investigated. In curved space-time, a non-relativistic particle is described by the Schrodinger equation [10]; therefore, the d-dimensional time-independent cosmic string equation will be reduced to a d-dimensional Schrodinger-like equation [23]. Laplacian equation and Hyperbolic Scarf plus Poschl-Teller and Manning-Rosen non-central potential are substituted into D-dimensional Schrodinger-like equation which is provided as

$$\left\{\begin{array}{l}-\dfrac{\hbar^2}{2M}\left(\begin{array}{l}\dfrac{1}{r^4}\dfrac{\partial}{\partial r}r^4\dfrac{\partial}{\partial r}+\dfrac{1}{r^2\sin^3\theta_4}\dfrac{\partial}{\partial\theta_4}\sin^3\theta_4\dfrac{\partial}{\partial\theta_4}\\+\dfrac{1}{r^2\sin^2\theta_4}\dfrac{1}{\sin^2\theta_3}\dfrac{\partial}{\partial\theta_3}\sin^2\theta_3\dfrac{\partial}{\partial\theta_3}\\+\dfrac{1}{r^2\sin^2\theta_3\sin^2\theta_4}\dfrac{1}{\sin\theta_2}\dfrac{\partial}{\partial\theta_2}\sin\theta_2\dfrac{\partial}{\partial\theta_2}\\+\dfrac{1}{r^2\alpha^2\sin^2\theta_2\sin^2\theta_3\sin^2\theta_4}\dfrac{\partial^2}{\partial\theta_1^2}\end{array}\right)\Psi\\+\dfrac{\hbar^2}{2M}\left[\begin{array}{l}\gamma^2\left[\dfrac{b^2+a(a+1)}{\sinh^2(\gamma r)}-\dfrac{2b\left(a+\frac{1}{2}\right)\cosh(\gamma r)}{\sinh^2(\gamma r)}\right]\\+\dfrac{1}{r^2}\left[\dfrac{\kappa_4(\kappa_4-1)}{\sin^2\theta_4}+\dfrac{\lambda_4(\lambda_4-1)}{\cos^2\theta_4}\right]\\+\dfrac{1}{r^2}\dfrac{1}{\sin^2\theta_4}\left[\dfrac{\kappa_3(\kappa_3-1)}{\sin^2\theta_3}+\dfrac{\lambda_3(\lambda_3-1)}{\cos^2\theta_3}\right]\\+\dfrac{1}{r^2}\dfrac{1}{\sin^2\theta_3\sin^2\theta_4}\left[\begin{array}{l}\dfrac{\nu_2(\nu_2+1)}{\sin^2\theta_2}\\-2\mu_2\cot\theta_2\end{array}\right]\\+\dfrac{1}{r^2}\dfrac{1}{\sin^2\theta_2\sin^2\theta_3\sin^2\theta_4}\left[\begin{array}{l}\dfrac{\nu_1(\nu_1+1)}{\cos^2\theta_1}\\-2\mu_1\tan\theta_1\end{array}\right]\end{array}\right]\Psi\end{array}\right\}=E\Psi \quad (10)$$

Equation (10) is a five-dimensional Schrodinger-like equation. The variable separation method is applied to the five-dimensional Schrodinger-like equation to obtain radial and angular part equations, where each part is a one-dimensional Schrodinger-like equation. By applying that method, we obtain equation (11) for the radial part, equation (12) for the θ_4 angular part, equation (13) for the θ_3 angular part, equation (14) for the θ_2 angular part, and equations (15) for the θ_1 angular part.

$$\left(\begin{array}{l}\dfrac{r^2}{R(r)r^4}\dfrac{\partial}{\partial r}r^4\dfrac{\partial}{\partial r}R(r)\\-\gamma^2r^2\left[\begin{array}{l}\dfrac{b^2+a(a+1)}{\sinh^2(\gamma r)}\\-\dfrac{2b\left(a+\frac{1}{2}\right)\cosh(\gamma r)}{\sinh^2(\gamma r)}\end{array}\right]+\dfrac{2M}{\hbar^2}r^2E\end{array}\right)=\lambda_4'' \quad (11)$$

$$\left(\begin{array}{l}\dfrac{1}{\Theta_4(\theta_4)\sin^3\theta_4}\dfrac{\partial}{\partial\theta_4}\sin^3\theta_4\dfrac{\partial}{\partial\theta_4}\Theta_4(\theta_4)\\-\left[\dfrac{\kappa_4(\kappa_4-1)}{\sin^2\theta_4}+\dfrac{\lambda_4(\lambda_4-1)}{\cos^2\theta_4}\right]-\dfrac{\lambda_3''}{\sin^2\theta_4}\end{array}\right)=-\lambda_4'' \quad (12)$$

$$\left(\begin{array}{l}\dfrac{1}{\Theta_3(\theta_3)}\dfrac{1}{\sin^2\theta_3}\dfrac{\partial}{\partial\theta_3}\sin^2\theta_3\dfrac{\partial}{\partial\theta_3}\Theta_3(\theta_3)\\-\left[\dfrac{\kappa_3(\kappa_3-1)}{\sin^2\theta_3}+\dfrac{\lambda_3(\lambda_3-1)}{\cos^2\theta_3}\right]-\dfrac{\lambda_2''}{\sin^2\theta_3}\end{array}\right)=-\lambda_3'' \quad (13)$$

$$\left(\begin{array}{l}\dfrac{1}{\Theta_2(\theta_2)}\dfrac{1}{\sin\theta_2}\dfrac{\partial}{\partial\theta_2}\sin\theta_2\dfrac{\partial}{\partial\theta_2}\Theta_2(\theta_2)\\-\left[\dfrac{\nu_2(\nu_2+1)}{\sin^2\theta_2}-2\mu_2\cot\theta_2\right]-\dfrac{\lambda_1''}{\alpha^2\sin^2\theta_2}\end{array}\right)=-\lambda_2'' \quad (14)$$

$$\frac{1}{\Theta_1(\theta_1)}\frac{\partial^2}{\partial\theta_1^2}\Theta_1(\theta_1)-\alpha^2\left[\frac{\nu_1(\nu_1+1)}{\cos^2\theta_1}-2\mu_1\tan\theta_1\right]=-\lambda_1'' \quad (15)$$

2.3. Hypergeometric Method

The hypergeometric method is applied to each one-dimensional Schrödinger-like equation to obtain the wave function and energy as the solution of d-dimensional time-independent cosmic string. The second-order equation for the hypergeometric function is written as

$$z(1-z)\frac{\partial^2\phi}{\partial z^2}+\left(c-(a+b+1)z\right)\frac{\partial\phi}{\partial z}-ab\phi=0 \quad (16)$$

Equation (16) includes two regular singular points: point $z=0$ and point $z=1$, but the solution with the point $z=0$ is simpler than that with the point $z=1$. Firstly, the solution considering point z = 0 is chosen, where the series formulation is given by

$$\phi=z^s\sum_{n=0}^{n}a_nz^n \quad (17)$$

The formulation for the hypergeometric equation's solution is provided in equation (18).

$$\begin{aligned}{}_2F_1(a,b,c;z)=\Phi(z)&=\sum_{n=0}\frac{(a)_n(b)_n}{(1)_n(c)_n}z^n\\&=\sum_{n=0}\frac{(a)_n(b)_n}{n!(c)_n}z^n\end{aligned} \quad (18)$$

where $(a)_n=a(a+1)(a+2)(a+3)...(a+n-1)$ and $(a)_0=1$.

The solution of the differential hypergeometric function, given in equation (18), has a value of all the denominators from the series that are not zero. Then, $c\neq -n$, where $n=0,1,2,3,4,...$. If $a=-n$ or $b=-n$, then the formulation of the solution is a series that becomes disconnected, so a finite solution is obtained, i.e., the polynomial rank of n; thus, the energy level of the system can be obtained.

3. Results and Discussion

3.1. Angular Part Solution of D-Dimensional Time-Independent Cosmic String for Hyperbolic Scarf Plus Poschl–Teller and Manning–Rosen Non-Central Potentials

To solve the θ_1 angular part, we assume that $\Theta_1(\theta_1)=H_1(\theta_1)$ in equation (15); we then obtain

$$\frac{\partial^2}{\partial\theta_1^2}H_1(\theta_1)-\left[\begin{array}{l}\dfrac{\nu_1'(\nu_1'+1)}{\cos^2\theta_1}\\-2\mu_1\alpha^2\tan\theta_1\end{array}\right]H_1(\theta_1)=-\lambda_1''H_1(\theta_1) \quad (19)$$

where $\nu_1'=\pm\sqrt{\alpha^2\nu_1(\nu_1+1)+\frac{1}{4}}-\frac{1}{2}$. For this solution, we assume that

$$\begin{aligned}\tan\theta_1&=i(1-2z)\\-\lambda_1''&=-k^2\end{aligned} \quad (20\text{–}21)$$

By substituting equations (20) and (21) in equation (19), we get

$$\begin{pmatrix} \left[z(1-z)\frac{d^2}{dz^2}+(1-2z)\frac{d}{dz}\right]H_1(\theta_1) \\ +\left[\nu_1^{'}\left(\nu_1^{'}+1\right)-\frac{2\mu_1\alpha^2 i+k^2}{4z}-\frac{-2\mu_1\alpha^2 i+k^2}{4(1-z)}\right]H_1(\theta_1) \end{pmatrix}=0 \quad (22)$$

For solution using $z=0$, equation (22) becomes equation (23). Because $\nu_1^{'}\left(\nu_1^{'}+1\right)$ and $-\frac{-2\mu_1\alpha^2 i+k^2}{4(1-z)}$ are ignored,

$$\begin{pmatrix} z(1-z)\frac{d^2}{dz^2}H_1(\theta_1) \\ +(1-2z)\frac{d}{dz}H_1(\theta_1)-\frac{2\mu_1\alpha^2 i+k^2}{4z}H_1(\theta_1) \end{pmatrix}=0 \quad (23)$$

The series as a solution for equation (23) is

$$H_1(\theta_1)=z^s\sum_{n=0}^{\infty}c_n z^n \quad (24)$$

Thus, we obtain

$$s=\sigma=\sqrt{\frac{2\mu_1\alpha^2 i+k^2}{4}} \quad (25)$$

For solution with $z=1$, equation (22) becomes equation (26). Because $\nu_1^{'}\left(\nu_1^{'}+1\right)$ and $-\frac{2\mu_1\alpha^2 i+k^2}{4z}$ are ignored,

$$\begin{pmatrix} z(1-z)\frac{d^2}{dz^2}H_1(\theta_1)+(1-2z)\frac{d}{dz}H_1(\theta_1) \\ -\frac{-2\mu_1\alpha^2 i+k^2}{4(1-z)}H_1(\theta_1) \end{pmatrix}=0 \quad (26)$$

The series as a solution for equation (26), which considers $z=1$ is

$$H_1(\theta_1)=(1-z)^p\sum_{m=0}D_m(1-z)^m \quad (27)$$

Thus, we can also determine

$$p=\beta=\sqrt{\frac{-2\mu_1\alpha^2 i+k^2}{4}} \quad (28)$$

From equation (24) and equation (27), we can write

$$H_1(\theta_1)=z^\sigma(1-z)^\beta f(z) \quad (29)$$

Then, we substitute equation (29) in equation (22) to obtain the following formulation

$$z(1-z)\begin{pmatrix} \sigma(\sigma-1)z^{\sigma-2}(1-z)^\beta f(z) \\ -\sigma z^{\sigma-1}\beta(1-z)^{\beta-1}f(z) \\ -\sigma z^{\sigma-1}\beta(1-z)^{\beta-1}f(z) \\ +z^\sigma\beta(\beta-1)(1-z)^{\beta-2}f(z) \\ +\sigma z^{\sigma-1}(1-z)^\beta f^{'}(z)-z^\sigma\beta(1-z)^{\beta-1}f^{'}(z) \\ +\sigma z^{\sigma-1}(1-z)^\beta f^{'}(z)-z^\sigma\beta(1-z)^{\beta-1}f^{'}(z) \\ +z^\sigma(1-z)^\beta f^{''}(z) \end{pmatrix}$$
$$+(1-2z)\begin{pmatrix} \sigma z^{\sigma-1}(1-z)^\beta f(z)-z^\sigma\beta(1-z)^{\beta-1}f(z) \\ +z^\sigma(1-z)^\beta f^{'}(z) \end{pmatrix}$$
$$+\left[\nu_2^{'}\left(\nu_2^{'}+1\right)-\frac{2\mu_1\alpha^2 i+k^2}{4z}-\frac{-2\mu_1\alpha^2 i+k^2}{4(1-z)}\right]$$
$$z^\sigma(1-z)^\beta f(z)=0 \quad (30)$$

The second-order equation for the hypergeometric function for the θ_1 angular part is written as

$$\begin{pmatrix} z(1-z)f^{''}(z)+\left[(2\sigma+1)-(2\sigma+2\beta+2)z\right]f^{'}(z) \\ -\left[-\nu_2^{'}\left(\nu_2^{'}+1\right)+(\sigma+\beta)(\sigma+\beta+1)\right]f(z) \end{pmatrix}=0 \quad (31)$$

From equation (31), we can obtain $a^{'}=\alpha+\beta-\nu^{'}$, $b^{'}=\alpha+\beta+1+\nu^{'}$ and $c^{'}=2\sigma+1$. We use $a^{'}=-n$, so we obtain

$$k^2=\left(\nu^{'}-n\right)^2-\frac{\mu_2^2\alpha^4}{\left(\nu^{'}-n\right)^2} \quad (32)$$

By using equation (21) and equation (32), we obtain the value of $\lambda_1^{''}$

$$\lambda_1^{''}=\left(\nu^{'}-n\right)^2-\frac{\mu_2^2\alpha^4}{\left(\nu^{'}-n\right)^2} \quad (33)$$

The ground state wave function for the θ_1 angular part

$$H_1(\theta_1)=\left(\frac{\tan\theta_1-i}{-2i}\right)^\sigma\left(1-\frac{\tan\theta_1-i}{-2i}\right)^\beta f(z) \quad (34)$$

To solve the θ_2 angular part, we assume $\Theta_2(\theta_2)=\frac{H_2(\theta_2)}{\sin^{1/2}\theta_2}$ for equation (14), and we can get

$$\begin{pmatrix} \frac{d^2}{d\theta_2^2}H_2(\theta_2) \\ -\begin{bmatrix} \frac{\nu_2^{'}\left(\nu_2^{'}+1\right)}{\sin^2\theta_2} \\ -2\mu_2\cot\theta_2 \end{bmatrix}H_2(\theta_2) \end{pmatrix}=-\left(\lambda_2^{''}+\frac{1}{4}\right)H_2(\theta_2) \quad (35)$$

where $\nu_2^{'}=\pm\sqrt{\nu_2(\nu_2+1)+\frac{\lambda_1^{''}}{\alpha^2}}-\frac{1}{2}$. Then we assume that

$$\cot\theta_2=i(1-2z) \quad (36)$$

$$-\left(\lambda_2^{''}+\frac{1}{4}\right)=-k^2 \quad (37)$$

$$\begin{pmatrix} \left[z(1-z)\frac{d^2}{dz^2}+(1-2z)\frac{d}{dz}\right]H_2(\theta_2) \\ +\begin{bmatrix} \nu_2^{'}\left(\nu_2^{'}+1\right)-\frac{-2\mu_2 i+k^2}{4z} \\ -\frac{2\mu_2 i+k^2}{4(1-z)} \end{bmatrix}H_2(\theta_2) \end{pmatrix}=0 \quad (38)$$

We applied the same method as the solution in the θ_1 angular part.

For solution around $z=0$, equation (38) becomes equation (39). Because $\nu_1^{'}\left(\nu_1^{'}+1\right)$ and $-\frac{2\mu_2 i+k^2}{4(1-z)}$ are ignored,

$$\begin{pmatrix} z(1-z)\frac{d^2}{dz^2}H_2(\theta_2)+(1-2z)\frac{d}{dz}H_2(\theta_2) \\ -\frac{-2\mu_2 i+k^2}{4z}H_2(\theta_2) \end{pmatrix}=0 \quad (39)$$

The series as a solution for equation (39) is

$$H_2(\theta_2)=z^s\sum_{n=0}^{\infty}c_n z^n \tag{40}$$

Then, we obtain

$$s=\sigma=\sqrt{\frac{-2\mu_2 i+k^2}{4}} \tag{41}$$

For solution around $z=1$, equation (38) becomes equation (42). Because $\nu_1^{'}(\nu_1^{'}+1)$ and $-\frac{-2\mu_2 i+k^2}{4z}$ are ignored,

$$\begin{pmatrix} z(1-z)\frac{d^2}{dz^2}H_2(\theta_2)+(1-2z)\frac{d}{dz}H_2(\theta_2) \\ -\frac{2\mu_2 i+k^2}{4(1-z)}H_2(\theta_2) \end{pmatrix}=0 \tag{42}$$

The series as a solution for equation (42), which is around $z=1$ is

$$H_1(\theta_1)=(1-z)^p\sum_{m=0}D_m(1-z)^m \tag{43}$$

Then, we also determine

$$p=\beta=\sqrt{\frac{2\mu_2 i+k^2}{4}} \tag{44}$$

from solution in the regular singular point $z=0$ and $z=1$. We obtain

$$H_2(\theta_2)=z^\sigma(1-z)^\beta f(z) \tag{45}$$

$$z(1-z)\begin{pmatrix} \sigma(\sigma-1)z^{\sigma-2}(1-z)^\beta f(z)-\sigma z^{\sigma-1}\beta(1-z)^{\beta-1} \\ f(z)-\sigma z^{\sigma-1}\beta(1-z)^{\beta-1}f(z) \\ +z^\sigma\beta(\beta-1)(1-z)^{\beta-2}f(z) \\ +\sigma z^{\sigma-1}(1-z)^\beta f^{'}(z)-z^\sigma\beta(1-z)^{\beta-1}f^{'}(z) \\ +\sigma z^{\sigma-1}(1-z)^\beta f^{'}(z)-z^\sigma\beta(1-z)^{\beta-1}f^{'}(z) \\ +z^\sigma(1-z)^\beta f^{''}(z) \end{pmatrix}$$
$$+(1-2z)\begin{pmatrix} \sigma z^{\sigma-1}(1-z)^\beta f(z)-z^\sigma\beta(1-z)^{\beta-1}f(z) \\ +z^\sigma(1-z)^\beta f^{'}(z) \end{pmatrix}$$
$$+\begin{bmatrix} \nu_2^{'}(\nu_2^{'}+1)-\frac{-2\mu_2 i+k^2}{4z} \\ -\frac{2\mu_2 i+k^2}{4(1-z)} \end{bmatrix}z^\sigma(1-z)^\beta f(z)=0 \tag{46}$$

The second-order equation for the θ_2 angular part is written as

$$\begin{pmatrix} z(1-z)f^{''}(z)+\begin{bmatrix}(2\sigma+1) \\ -(2\sigma+2\beta+2)z\end{bmatrix}f^{'}(z) \\ -\left[-\nu_2^{'}(\nu_2^{'}+1)+(\sigma+\beta)(\sigma+\beta+1)\right]f(z) \end{pmatrix}=0 \tag{47}$$

We obtain $a^{'}=\alpha+\beta-\nu^{'}$, $b^{'}=\alpha+\beta+1+\nu^{'}$ and $c^{'}=2\sigma+1$. We use $a^{'}=-n$, then obtain the value of k^2 in equation (48), and by using equations (35) and (37), we obtain the value of $\lambda_2^{''}$ in equation (49).

$$k^2=(\nu^{'}-n)^2-\frac{\mu_2^2}{(\nu^{'}-n)^2} \tag{48}$$

$$\lambda_2^{''}=(\nu^{'}-n)^2-\frac{\mu_2^2}{(\nu^{'}-n)^2}-\frac{1}{4} \tag{49}$$

Ground state wave function for θ_2 angular part

$$H_2(\theta_2)=\left(\frac{-\cot\theta_2+i}{i2}\right)^\sigma\left(1-\frac{-\cot\theta_2+i}{i2}\right)^\beta f(z) \tag{50}$$

To solve the θ_3 angular part, we assume $\Theta_3(\theta_3)=\frac{H_3(\theta_3)}{\sin\theta_3}$ for equation (13), and we can define

$$\frac{d^2}{d\theta_3^2}H_3(\theta_3)-\begin{bmatrix}\frac{\kappa_3^{'}(\kappa_3^{'}-1)}{\sin^2\theta_3} \\ +\frac{\lambda_3(\lambda_3-1)}{\cos^2\theta_3}\end{bmatrix}H_3(\theta_3)=-(\lambda_3^{''}+1)H_3(\theta_3) \tag{51}$$

where $\kappa_3^{'}=\pm\sqrt{\kappa_3(\kappa_3-1)+\lambda_2^{''}+\frac{1}{4}}+\frac{1}{2}$. We assume some equation to the equation (51)

$$\cos^2\theta_3=z \tag{52}$$

$$-(\lambda_3^{''}+1)=-k^2 \tag{53}$$

$$\begin{pmatrix} z(1-z)\frac{d^2}{dz^2}H_3(\theta_3)+\frac{1}{2}(1-2z)\frac{d}{dz}H_3(\theta_3) \\ +\left[\frac{k^2}{4}-\frac{\kappa_3^{'}(\kappa_3^{'}-1)}{4(1-z)}-\frac{\lambda_3(\lambda_3-1)}{4z}\right]H_3(\theta_3) \end{pmatrix}=0 \tag{54}$$

We applied the same method as the solution in the θ_1 angular part. For the solution around $z=0$, equation (54) becomes equation (55). Because $\frac{k^2}{4}$ and $-\frac{\kappa_3^{'}(\kappa_3^{'}-1)}{4(1-z)}$ are ignored,

$$\begin{pmatrix} z(1-z)\frac{d^2}{dz^2}H_3(\theta_3)+\frac{1}{2}(1-2z)\frac{d}{dz}H_3(\theta_3) \\ -\frac{\lambda_3(\lambda_3-1)}{4z}H_3(\theta_3) \end{pmatrix}=0 \tag{55}$$

The series as a solution for equation (55) is:

$$H_2(\theta_2)=z^s\sum_{n=0}^{\infty}c_n z^n \tag{56}$$

Then, we obtain

$$s=\sigma=\frac{\lambda_3}{2} \tag{57}$$

For the solution around $z=1$, equation (54) becomes equation (58). Because $\frac{k^2}{4}$ and $-\frac{\lambda_3(\lambda_3-1)}{4z}$ are ignored,

$$\begin{pmatrix} z(1-z)\frac{d^2}{dz^2}H_3(\theta_3)+\frac{1}{2}(1-2z)\frac{d}{dz}H_3(\theta_3) \\ -\frac{\kappa_3^{'}(\kappa_3^{'}-1)}{4(1-z)}H_3(\theta_3) \end{pmatrix}=0 \quad (58)$$

The series as a solution for equation (58), which is around $z=1$ is

$$H_1(\theta_1)=(1-z)^p\sum_{m=0}D_m(1-z)^m \quad (59)$$

Then, we also determine

$$p=\beta=\frac{\kappa_2^{'}}{2} \quad (60)$$

From solution in the regular singular point, we obtain

$$H_3(\theta_3)=z^{\sigma}(1-z)^{\beta}f(z) \quad (61)$$

$$z(1-z)\begin{pmatrix} \sigma(\sigma-1)z^{\sigma-2}(1-z)^{\beta}f(z)-\sigma z^{\sigma-1}\beta(1-z)^{\beta-1} \\ f(z)-\sigma z^{\sigma-1}\beta(1-z)^{\beta-1}f(z) \\ +z^{\sigma}\beta(\beta-1)(1-z)^{\beta-2}f(z) \\ +\sigma z^{\sigma-1}(1-z)^{\beta}f^{'}(z)-z^{\sigma}\beta(1-z)^{\beta-1}f^{'}(z) \\ +\sigma z^{\sigma-1}(1-z)^{\beta}f^{'}(z)-z^{\sigma}\beta(1-z)^{\beta-1}f^{'}(z) \\ +z^{\sigma}(1-z)^{\beta}f^{"}(z) \end{pmatrix}$$
$$+\frac{1}{2}(1-2z)\begin{pmatrix} \sigma z^{\sigma-1}(1-z)^{\beta}f(z)-z^{\sigma}\beta(1-z)^{\beta-1}f(z) \\ +z^{\sigma}(1-z)^{\beta}f^{'}(z) \end{pmatrix} \quad (62)$$
$$+\left[\frac{k^2}{4}-\frac{\kappa_3^{'}(\kappa_3^{'}-1)}{4(1-z)}-\frac{\lambda_3(\lambda_3-1)}{4z}\right]z^{\sigma}(1-z)^{\beta}f(z)=0$$

The second-order equation for the θ_3 angular part is written as

$$\begin{pmatrix} z(1-z)f^{"}(z)+\begin{bmatrix}\left(2\sigma+\frac{1}{2}\right) \\ -(2\sigma+2\beta+1)z\end{bmatrix}f^{'}(z) \\ -\left((\sigma+\beta)^2-\frac{k^2}{4}\right)f(z) \end{pmatrix}=0 \quad (63)$$

We obtain $a^{'}=\alpha+\beta-\frac{k}{2}$, $b^{'}=\alpha+\beta+\frac{k}{2}$ and $c^{'}=2\sigma+\frac{1}{2}$. We use $a^{'}=-n$, then we determine the value

$$k^2=\left(\lambda_3+\kappa_3^{'}+2n\right)^2 \quad (64)$$

$$\lambda_3^{"}=\left(\lambda_3+\kappa_3^{'}+2n\right)^2-1 \quad (65)$$

By using equations (53) and (64), we obtain the value of $\lambda_3^{"}$ in equation (65). The ground state wave function for the θ_3 angular part

$$H_3(\theta_3)=\cos^2\theta_3^{\frac{\lambda_3^{'}}{2}}\left(-\sin^2\theta_3\right)^{\frac{\kappa_3^{'}}{2}}f(z) \quad (66)$$

To solve other angular parts, we assume $\Theta_4(\theta_4)=\frac{H_4(\theta_4)}{\sin^{3/2}\theta_4}$ for equation (12), and then we define

$$\begin{pmatrix} \frac{d^2}{d\theta_4^2}H_4(\theta_4) \\ -\begin{bmatrix}\frac{\kappa_4^{'}(\kappa_4^{'}-1)}{\sin^2\theta_4} \\ +\frac{\lambda_4(\lambda_4-1)}{\cos^2\theta_4}\end{bmatrix}H_4(\theta_4) \end{pmatrix}=-\left(\lambda_4^{"}+\frac{3}{2}+\frac{3}{4}\right)H_4(\theta_4) \quad (67)$$

where $\kappa_4^{'}=\pm\sqrt{\kappa_4(\kappa_4-1)+\lambda_3^{"}+1}+\frac{1}{2}$. We assume that

$$\cos^2\theta_4=z \quad (68)$$

$$-\left(\lambda_4^{"}+\frac{3}{2}+\frac{3}{4}\right)=-k^2 \quad (69)$$

We also applied the same method as the θ_1 angular part. From solution in the point regular singular, we also obtain

$$\sigma=\frac{\lambda_4}{2} \text{ and } \beta=\frac{\kappa_4^{'}}{2} \quad (70)$$

$$H_4(\theta_4)=z^{\sigma}(1-z)^{\beta}f(z) \quad (71)$$

The second-order equation for the θ_4 angular part is written as

$$\begin{pmatrix} z(1-z)f^{"}(z)+\begin{bmatrix}\left(2\sigma+\frac{1}{2}\right) \\ -(2\sigma+2\beta+1)z\end{bmatrix}f^{'}(z) \\ -\left((\sigma+\beta)^2-\frac{k^2}{4}\right)f(z) \end{pmatrix}=0 \quad (72)$$

We use $a^{'}=-n$, then obtain the value of k^2 in equation (73). By using equations (69) and (73), we obtain the value of $\lambda_4^{"}$ in equation (74).

$$k^2=\left(\lambda_4+\kappa_4^{'}+2n\right)^2 \quad (73)$$

$$\lambda_4^{"}=\left(\lambda_4+\kappa_4^{'}+2n\right)^2-\frac{9}{4} \quad (74)$$

The ground state wave function for the θ_4 angular part

$$H_4(\theta_4)=\cos^2\theta_4^{\frac{\lambda_4^{'}}{2}}\left(-\sin^2\theta_4\right)^{\frac{\kappa_4^{'}}{2}}f(z) \quad (75)$$

3.2. Angular Part Solution of D-Dimensional Time-Independent Cosmic String for Hyperbolic Scarf Plus Poschl-Teller and Manning-Rosen Non-Central Potentials

To solve the radial part, we assume that $R(r)=\frac{X(r)}{r^2}$ and $\frac{1}{r^2}=\gamma^2\left(\frac{1}{\sin^2(\gamma r)}+d_0\right)$ to equation (11). Then, we can find

$$\frac{d^2}{dr^2}X(r)-\gamma^2\left[\begin{array}{c}\frac{b^2+a(a+1)+\lambda_4^{"}+2}{\sinh^2(\gamma r)}\\ -\frac{2b\left(a+\frac{1}{2}\right)\cosh(\gamma r)}{\sinh^2(\gamma r)}\end{array}\right]X(r)=\left[\begin{array}{c}-\frac{2M}{\hbar^2}E\\ +\gamma^2\left(\lambda_4^{"}+2\right)d_0\end{array}\right]X(r) \quad (76)$$

then, we also assume that

$$\cosh(\gamma r)=1-2z \quad (77)$$

$$\left[\frac{2M}{\hbar^2}E-\gamma^2\left(\lambda_4^{"}+2\right)d_0\right]=-k^2 \quad (78)$$

By substituting equations (77) and (78) in equation (76), we obtain

$$\left[\begin{array}{c}z(1-z)\frac{d^2}{dz^2}\\ +\frac{1}{2}(1-2z)\frac{d}{dz}\end{array}\right]X(r)+\left[\begin{array}{c}\frac{k^2}{\gamma^2}-\frac{\left(\left(a+\frac{1}{2}\right)-b\right)^2-\frac{1}{4}+\lambda_4^{"}+2}{4z}\\ -\frac{\left(\left(a+\frac{1}{2}\right)+b\right)^2-\frac{1}{4}+\lambda_4^{"}+2}{4(1-z)}\end{array}\right]X(r)=0 \quad (79)$$

For the solution around $z=0$, equation (79) becomes equation (80). Because $\frac{k^2}{\gamma^2}$ and $\frac{\left(\left(a+\frac{1}{2}\right)+b\right)^2-\frac{1}{4}+\lambda_4^{"}+2}{4(1-z)}$ are ignored,

$$\left(\begin{array}{c}z(1-z)\frac{d^2}{dz^2}X(r)+\frac{1}{2}(1-2z)\frac{d}{dz}X(r)\\ -\frac{\left(\left(a+\frac{1}{2}\right)-b\right)^2-\frac{1}{4}+\lambda_4^{"}+2}{4z}X(r)\end{array}\right)=0 \quad (80)$$

The series as the solution of the equation (80) is

$$X(r)=z^s\sum_{n=0}^{\infty}c_n z^n \quad (81)$$

Then, we can get

$$s=\sigma=\frac{\sqrt{\left(\left(a+\frac{1}{2}\right)-b\right)^2+\lambda_4^{"}+2}+\frac{1}{2}}{2} \quad (82)$$

For the solution around $z=1$, equation (79) becomes equation (83). Because $\frac{k^2}{\gamma^2}$ and $\frac{\left(\left(a+\frac{1}{2}\right)-b\right)^2-\frac{1}{4}+\lambda_4^{"}+2}{4z}$ are ignored,

$$\left(\begin{array}{c}z(1-z)\frac{d^2}{dz^2}X(r)+\frac{1}{2}(1-2z1)\frac{d}{dz}X(r)\\ -\frac{\left(\left(a+\frac{1}{2}\right)+b\right)^2-\frac{1}{4}+\lambda_4^{"}+2}{4(1-z)}X(r)\end{array}\right)=0 \quad (83)$$

The series as a solution for equation (83), which is around $z=1$ is

$$X(r)=(1-z)^p\sum_{m=0}D_m(1-z)^m \quad (84)$$

Then we obtain

$$p=\beta=\frac{-\sqrt{\left(\left(a+\frac{1}{2}\right)+b\right)^2+\lambda_4^{'}+2}+\frac{1}{2}}{2} \quad (85)$$

From equation (81) and equation (84), we can write

$$X=z^{\sigma}(1-z)^{\beta}f(z) \quad (86)$$

Then, we substitute equation (86) in equation (79) and obtain a formulation:

$$z(1-z)\left(\begin{array}{l}\sigma(\sigma-1)z^{\sigma-2}(1-z)^{\beta}f(z)-\sigma z^{\sigma-1}\beta(1-z)^{\beta-1}\\ f(z)-\sigma z^{\sigma-1}\beta(1-z)^{\beta-1}f(z)\\ +z^{\sigma}\beta(\beta-1)(1-z)^{\beta-2}f(z)\\ +\sigma z^{\sigma-1}(1-z)^{\beta}f^{'}(z)-z^{\sigma}\beta(1-z)^{\beta-1}f^{'}(z)\\ +\sigma z^{\sigma-1}(1-z)^{\beta}f^{'}(z)-z^{\sigma}\beta(1-z)^{\beta-1}f^{'}(z)\\ +z^{\sigma}(1-z)^{\beta}f^{"}(z)\end{array}\right)+\frac{1}{2}(1-2z1)\left(\begin{array}{l}\sigma z^{\sigma-1}(1-z)^{\beta}f(z)-z^{\sigma}\beta(1-z)^{\beta-1}f(z)\\ +z^{\sigma}(1-z)^{\beta}f^{'}(z)\end{array}\right)+\left(\begin{array}{c}\frac{k^2}{\gamma^2}-\frac{\left(\left(a+\frac{1}{2}\right)-b\right)^2-\frac{1}{4}+\lambda_4^{'}+2}{4z}\\ -\frac{\left(\left(a+\frac{1}{2}\right)+b\right)^2-\frac{1}{4}+\lambda_4^{'}+2}{4(1-z)}\end{array}\right)z^{\sigma}(1-z)^{\beta}f(z)=0 \quad (87)$$

The second-order equation for the hypergeometric function for the radial part is written as

$$\left(\begin{array}{c}z(1-z)f^{"}(z)+\left[\begin{array}{c}\left(2\sigma+\frac{1}{2}\right)\\ -(2\sigma+2\beta+1)z\end{array}\right]f^{'}(z)\\ -\left((\sigma+\beta)^2-\frac{k^2}{\gamma^2}\right)f(z)\end{array}\right)=0 \quad (88)$$

From equation (88), we can determine $a^{'}=\sigma+\beta+\frac{k}{\gamma}$, $b^{'}=\sigma+\beta-\frac{k}{\gamma}$ and $c^{'}=2\sigma+\frac{1}{2}$. We also use $a^{'}=-n$, then we obtain

$$k^2=\gamma^2(-\sigma-\beta-n)^2 \quad (89)$$

By using equations (78) and (89), the energy equation is solved, and that is written in equation (90).

$$E=-\frac{\hbar^2\gamma^2}{2M}\left[(-\sigma-\beta-n)^2-\left(\lambda_4^{"}+2\right)d_0\right] \quad (90)$$

Energy levels of d-dimension time-independent cosmic string for Hyperbolic Scarf plus Poschl-Teller and Manning-Rosen non-central potential with Quantum number were shown in Table 1. Increasing the quantum number n_{θ_1}, n_{θ_2}, n_{θ_3}, and n_{θ_4} caused the increase in energy level. However, increasing quantum number n_r caused decreasing in energy level. Potential parameters are also associated with energy levels. Value of parameters caused significant effects on energy levels. Table 2 showed that the increase in potential parameters caused increasing in energy levels. The increasing value of all potential parameters only caused increasing in energy levels. Decreasing parameter α also caused decreasing in energy levels. However, in the variation $\alpha=0.3$, energy becomes large and decreases again. So, the plot of energy levels was a function of the quartic. If the value of parameter cosmic string $\alpha = 1$, the equation and energy are like Schrodinger equation without cosmic string.

Table 1 Energy levels of d-dimensional time-independent cosmic string for Hyperbolic Scarf plus Poschl-Teller and Manning-Rosen non-central potentials with quantum number variations ($\gamma = 0.05$, $a = 3$, $b = 2$, $\kappa_3 = \kappa_4 = \lambda_3 = \lambda_4 = 1.5$, $\nu_1 = \nu_2 = 4$, $\mu_1 = \mu_2 = 2$)

n_r	n_{θ_1}	n_{θ_2}	n_{θ_3}	n_{θ_4}	Energy				
					$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.6$	$\alpha = 0.3$	$\alpha = 0.1$
0	1	1	1	1	1.2805E-02	1.1748E-02	9.5591E-03	1.4517E-01	1.1623E-02
1	1	1	1	1	1.1790E-02	1.0790E-02	8.7436E-03	1.4313E-01	1.0672E-02
2	1	1	1	1	8.2747E-03	7.3320E-03	5.4281E-03	1.3860E-01	7.2212E-03
3	1	1	1	1	2.2594E-03	1.3739E-03	-3.8734E-04	1.0082E-01	1.2703E-03
4	1	1	1	1	-6.2559E-03	-7.0842E-03	-8.7028E-03	1.2203E-01	-7.1806E-03
5	1	1	1	1	-1.7271E-02	-1.8042E-02	-1.9518E-02	1.1000E-01	-1.8131E-02
1	0	1	1	1	1.4117E-02	1.3064E-02	1.0043E-02	1.0076E-02	3.9083E-02
1	2	1	1	1	9.4886E-03	8.1755E-03	2.7923E-02	9.4200E-03	9.0411E-03
1	3	1	1	1	9.7333E-03	1.3320E-02	1.1077E-02	9.6574E-03	1.1405E-02
1	4	1	1	1	-	1.5492E-02	8.8758E-03	1.1944E-02	1.3708E-02
1	5	1	1	1	9.7333E-03	7.8371E-03	1.1273E-02	1.4282E-02	1.6185E-02
1	1	0	1	1	1.1790E-02	1.0790E-02	8.7436E-03	1.4313E-01	1.0672E-02
1	1	2	1	1	1.1790E-02	1.0790E-02	8.7436E-03	1.4313E-01	1.0672E-02
1	1	3	1	1	1.1790E-02	1.0790E-02	8.7436E-03	1.4313E-01	1.0672E-02
1	1	4	1	1	1.1790E-02	1.0790E-02	8.7436E-03	1.4313E-01	1.0672E-02
1	1	5	1	1	1.1790E-02	1.0790E-02	8.7436E-03	1.4313E-01	1.0672E-02
1	1	1	0	1	7.8547E-03	7.0627E-03	5.4805E-03	1.2803E-01	6.9700E-03
1	1	1	2	1	1.6607E-02	1.5402E-02	1.2898E-02	1.5908E-01	1.5259E-02
1	1	1	3	1	2.2294E-02	2.0886E-02	1.7930E-02	1.7586E-01	2.0719E-02
1	1	1	4	1	2.8844E-02	2.7235E-02	2.3832E-02	1.9347E-01	2.7043E-02
1	1	1	5	1	3.6250E-02	3.4442E-02	3.0593E-02	2.1192E-01	3.4225E-02
1	1	1	1	0	7.8181E-03	7.0220E-03	5.4270E-03	1.2802E-01	6.9288E-03
1	1	1	1	2	1.6638E-02	1.5435E-02	1.2939E-02	1.5908E-01	1.5293E-02
1	1	1	1	3	2.2353E-02	2.0950E-02	1.8007E-02	1.7587E-01	2.0783E-02
1	1	1	1	4	2.8929E-02	2.7327E-02	2.3942E-02	1.9349E-01	2.7136E-02
1	1	1	1	5	3.6361E-02	3.4560E-02	3.0736E-02	2.1194E-01	3.4345E-02

Table 2 Energy levels of d-dimensional time-independent cosmic string for Hyperbolic Scarf plus Poschl-Teller and Manning-Rosen non-central potentials with the parameters' potential variations ($n_r = n_{\theta_1} = n_{\theta_2} = n_{\theta_3} = n_{\theta_4} = 1$, $\mu_1 = \mu_2 = 2$, $\gamma = 0.05$)

a	b	ν_1	ν_2	κ_3	κ_4	λ_3	λ_4	Energy				
								$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.6$	$\alpha = 0.3$	$\alpha = 0.1$
3	2	4	4	1.5	1.5	1.5	1.5	1.1790E-02	1.0790E-02	8.7436E-03	1.4313E-01	1.0672E-02
4	2	4	4	1.5	1.5	1.5	1.5	1.2100E-02	1.1099E-02	9.0471E-03	1.4330E-01	1.0981E-02
5	2	4	4	1.5	1.5	1.5	1.5	1.2335E-02	1.1329E-02	9.2618E-03	1.4346E-01	1.1210E-02
3	3	4	4	1.5	1.5	1.5	1.5	1.2330E-02	1.1329E-02	9.2723E-03	1.4343E-01	1.1211E-02
3	4	4	4	1.5	1.5	1.5	1.5	1.2657E-02	1.1640E-02	9.5390E-03	1.4370E-01	1.1520E-02
3	2	5	4	1.5	1.5	1.5	1.5	1.4117E-02	1.2829E-02	9.0273E-03	2.9574E-02	1.1168E-02
3	2	6	4	1.5	1.5	1.5	1.5	1.6628E-02	1.4983E-02	1.0483E-02	1.4124E-02	1.1824E-02
3	2	4	5	1.5	1.5	1.5	1.5	1.1790E-02	1.0790E-02	8.7436E-03	1.4313E-01	1.0672E-02
3	2	4	6	1.5	1.5	1.5	1.5	1.1790E-02	1.0790E-02	8.7436E-03	1.4313E-01	1.0672E-02
3	2	4	4	2	1.5	1.5	1.5	1.2229E-02	1.1279E-02	9.4303E-03	1.4330E-01	1.1168E-02
3	2	4	4	2.5	1.5	1.5	1.5	1.2814E-02	1.1920E-02	1.0251E-02	1.4353E-01	1.1817E-02
3	2	4	4	1.5	2	1.5	1.5	1.1984E-02	1.0988E-02	8.9521E-03	1.4328E-01	1.0870E-02
3	2	4	4	1.5	2.5	1.5	1.5	1.2254E-02	1.1263E-02	9.2417E-03	1.4348E-01	1.1146E-02
3	2	4	4	1.5	1.5	2	1.5	1.2912E-02	1.1861E-02	9.6991E-03	1.4704E-01	1.1736E-02
3	2	4	4	1.5	1.5	2.5	1.5	1.4089E-02	1.2986E-02	1.0710E-02	1.5100E-01	1.2856E-02
3	2	2	2	1.5	1.5	1.5	2	1.2920E-02	1.1869E-02	9.7101E-03	1.4704E-01	1.1745E-02
3	2	2	2	1.5	1.5	1.5	2.5	1.4105E-02	1.3003E-02	1.0731E-02	1.5100E-01	1.2873E-02

Ground state energy for radial part

$$X(r) = z^{\sigma}(1-z)^{\beta} F_1\left(a^{'}, b^{'}; c^{'}, z\right) \quad (91)$$

By substituting equations (77), (82), and (85) in equation (91), we can find

$$X(r) = \left(\begin{array}{c} \left(\frac{1-\cosh(\gamma r)}{2}\right)^{\sigma} \left(\frac{1+\cosh(\gamma r)}{2}\right)^{\beta} \\ {}_2F_1\left(-n, 2\sigma + 2\beta + n; \left(2\sigma + \frac{1}{2}\right), z\right) \end{array} \right) \quad (92)$$

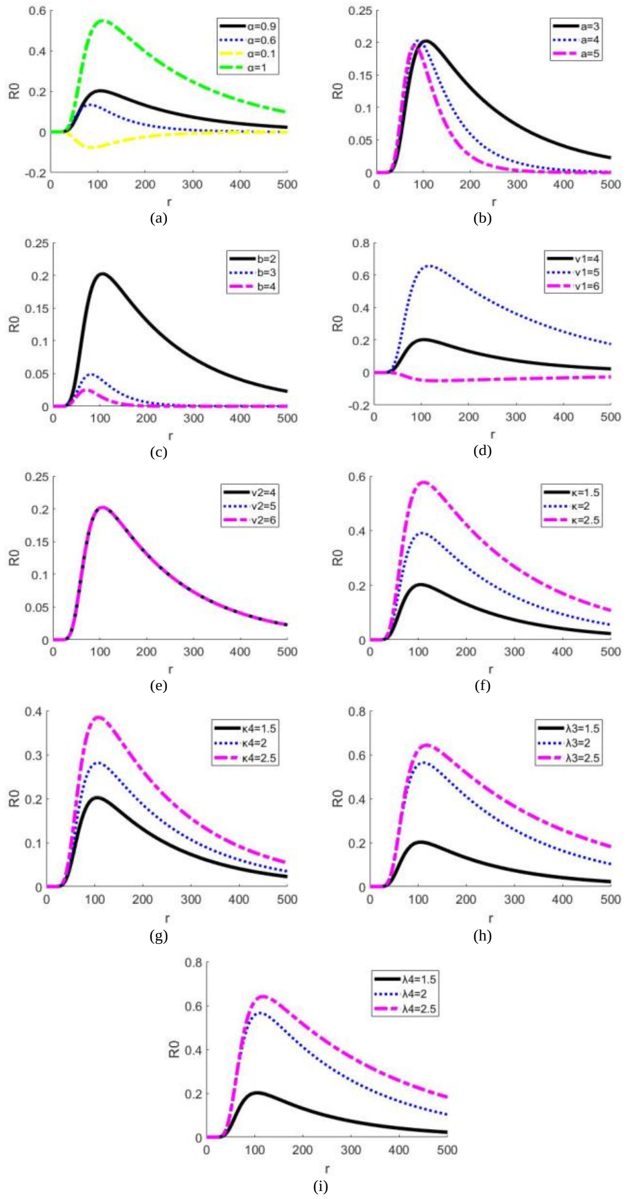

Fig. 1 Ground state wave function for radial part with (a) α variation, (b) a variation, (c) b variation, (d) ν_1 variation, (e) ν_2 variation, (f) κ_3 variation, (g) κ_4 variation, (h) λ_3 variation and (i) λ_4 variation

Fig. 1 showed that the wave amplitude of the ground state wave function for the radial part was variated. That is caused by variation of parameters a , b , ν_1, ν_2, κ_3, κ_4, λ_3, λ_4, α and variation of n . Variation of the value of parameters caused variation of waveforms. If the value of the parameter a increased, then the amplitude of wave function decreased. If the value of the parameter b increased, the amplitude of wave function decreased. If the value of the parameter ν_1 increased, then the amplitude of wave function decreased. If the value of the parameter ν_2 increases, then the amplitude of wave function is still in the same value. If the value of the parameter κ_3 increased, then the amplitude of wave function increased. If the value of the parameter κ_4 increased, the amplitude of wave function increased. If the value of the parameter λ_3 increased, the amplitude of wave function increased. If the value of the parameter λ_4 increased, the amplitude of wave function increased. If the value of the parameter α decreased, the amplitude of wave function decreased.

Renyi entropy is defined by equation (93).

$$R_q(\rho) = \frac{1}{1-q}\ln 4\pi\int_0^\infty \rho(r)^q\, dr \tag{93}$$

By using the ground state wave function in equation (92), $\rho(r)$ is written as

$$\rho(r) = z^{2\sigma}(1-z)^{2\beta}\ {}_2F_1\begin{pmatrix} -n, 2\sigma+2\beta+n; \\ \left(2\sigma+\frac{1}{2}\right), z \end{pmatrix}^2 \tag{94}$$

where

$$\cosh(\gamma r) = 1-2z$$
$$dr = \frac{-dz}{\gamma\sqrt{z(z-1)}} \tag{95}$$
$$dr = \frac{dz}{\gamma\sqrt{z(1-z)}}$$

By using equation (95), equation (93) becomes

$$R(\rho) = \frac{1}{1-q}\ln 4\pi\int_0^1 \rho(z)^q\, \frac{dz}{\gamma\sqrt{z(1-z)}} \tag{96}$$

$$R(\rho) = \frac{1}{1-q}\left(\ln 4\pi + q\ln\int_0^1 \rho(z)\frac{dz}{\gamma\sqrt{z(1-z)}}\right) \tag{97}$$

Then, we substitute equation (94) into equation (97) to obtain

$$R(\rho) = \frac{1}{1-q}\begin{pmatrix} \ln 4\pi \\ +q\ln\frac{1}{\gamma}\int_0^1 \begin{pmatrix} z^{2\sigma-\frac{1}{2}}(1-z)^{2\beta-\frac{1}{2}} \\ {}_2F_1\begin{pmatrix} -n, 2\sigma+2\beta+n \\ ;2\sigma+\frac{1}{2}, z\end{pmatrix}^2 \end{pmatrix} dz \end{pmatrix} \tag{98}$$

If

$$\int_0^1 z^{2\sigma-\frac{1}{2}}(1-z)^{2\beta-\frac{1}{2}}\ {}_2F_1\left(-n, 2\sigma+2\beta+n; 2\sigma+\frac{1}{2}, z\right)^2 dz$$
$$= \frac{n!(-2\sigma-2\beta+1-2n)_n\,\Gamma\left(2\sigma+\frac{1}{2}\right)\Gamma\left(2\beta+\frac{1}{2}+n\right)}{(a)_n\,\Gamma(2\sigma+2\beta+1+2n)} \tag{99}$$

by substituting equation (99) into equation (98), we then define

$$R(\rho) = \frac{1}{1-q}\left(\ln 4\pi + q\ln\frac{1}{\gamma}\frac{\left(\begin{matrix} n!(-2\sigma-2\beta+1-2n)_n \\ \Gamma\left(2\sigma+\frac{1}{2}\right)\Gamma\left(2\beta+\frac{1}{2}+n\right)\end{matrix}\right)}{(a)_n\,\Gamma(2\sigma+2\beta+1+2n)}\right) \tag{100}$$

$$R(\rho) = \frac{1}{1-q}\ln 4\pi\left(\frac{1}{\gamma}\right)^q\left(\frac{\left(\begin{matrix} n!(-2\sigma-2\beta+1-2n)_n \\ \Gamma\left(2\sigma+\frac{1}{2}\right)\Gamma\left(2\beta+\frac{1}{2}+n\right)\end{matrix}\right)}{(a)_n\,\Gamma(2\sigma+2\beta+1+2n)}\right)^q \tag{101}$$

Equation (101) is the Renyi entropy with $q > 0$, $q \neq 1$.

The Renyi entropy can be used to calculate the mass-energy parameter of a black hole and the temperature of a black hole. The Renyi entropy for a black hole is written as

$$S_R = \frac{1}{1-q}\ln\left(1+(1-q)S_{BH}\right) \tag{102}$$

where $S_{BH} = 4\pi M^2$ so equation (102) becomes

$$S_R = \frac{1}{1-q}\ln\left(1+4\pi M^2(1-q)\right) \tag{103}$$

and M is written as

$$M = \frac{1}{2\pi(1-q)}\sqrt{\begin{matrix} 4\pi\left(\frac{1}{\gamma}\right)^q \\ (1-q)\left(\frac{\left(\begin{matrix} n!(-2\sigma-2\beta+1-2n)_n \\ \Gamma\left(2\sigma+\frac{1}{2}\right)\Gamma\left(2\beta+\frac{1}{2}+n\right)\end{matrix}\right)}{(a)_n\,\Gamma(2\sigma+2\beta+1+2n)}\right)^q \\ -\pi(1-q)\end{matrix}} \tag{104}$$

M is the mass-energy parameter of the black hole and S_{BH} is the Bekenstein-Hawking entropy. So, equation (104) is the mass of the Schwarzschild black hole solution.

The temperature of the black hole for Renyi entropy is written as

$$T_R = \frac{1}{8\pi M} + \frac{(1-q)M}{2} \tag{105}$$

By substituting equation (104) into equation (105), the temperature for a black hole is obtained. This is written in equation (106).

$$T_R = \frac{\left(\frac{1}{\gamma}\right)^q \left(\frac{n!\left(-2\sigma-2\beta+1-2n\right)_n}{\frac{\Gamma\left(2\sigma+\frac{1}{2}\right)\Gamma\left(2\beta+\frac{1}{2}+n\right)}{(a)_n\Gamma\left(2\sigma+2\beta+1+2n\right)}}\right)^q}{\left(\frac{1}{(1-q)}\sqrt{4\pi\left(\frac{1}{\gamma}\right)^q(1-q)\left(\frac{n!\left(-2\sigma-2\beta+1-2n\right)_n}{\frac{\Gamma\left(2\sigma+\frac{1}{2}\right)\Gamma\left(2\beta+\frac{1}{2}+n\right)}{(a)_n\Gamma\left(2\sigma+2\beta+1+2n\right)}}\right)^q - \pi(1-q)}\right)} \quad (106)$$

The heat capacity for a black hole is written as

$$C_R = \frac{8\pi M^2}{4\pi\left(1-q'\right)M^2-1} \quad (107)$$

$$C_R = \frac{\frac{1}{(1-q)}\left(4\left(\frac{1}{\gamma}\right)^q\left(\frac{n!\left(-2\sigma-2\beta+1-2n\right)_n}{\frac{\Gamma\left(2\sigma+\frac{1}{2}\right)\Gamma\left(2\beta+\frac{1}{2}+n\right)}{(a)_n\Gamma\left(2\sigma+2\beta+1+2n\right)}}\right)^q-1\right)}{\left(\left(\frac{1}{\gamma}\right)^q\left(\frac{n!\left(-2\sigma-2\beta+1-2n\right)_n}{\frac{\Gamma\left(2\sigma+\frac{1}{2}\right)\Gamma\left(2\beta+\frac{1}{2}+n\right)}{(a)_n\Gamma\left(2\sigma+2\beta+1+2n\right)}}\right)^q-1\right)-1} \quad (108)$$

Equation (104) is substituted into equation (107) to obtain heat capacity. Heat capacity is written in equation (108).

4. Conclusion

Wave function and energy spectra for a time-independent cosmic string in d-dimension with Hyperbolic Scarf plus Poschl-Teller and Manning-Rosen non-central potentials have been investigated in this research. Energy levels of d-dimensional time-independent cosmic string with Hyperbolic Scarf plus Poschl-Teller and Manning-Rosen non-central potentials were decreased by the presence of potentials parameters. Energy levels were higher with potentials parameters equal to zero. Energy levels increased if the value of potentials parameters was increased and if the values of the numbers n_{θ_1}, n_{θ_2}, n_{θ_3}, and n_{θ_4} were increased. Energy levels decreased if the value of the parameter α was decreased. However, the variation of $\alpha = 0.3$ caused the energy to increase and then decrease again. The plot of energy levels was a function of the quartic. The increasing value of the number n_r caused a decrease in energy levels. In this work, the energy eigenvalue and wave equation were applied to the study of Renyi entropy and the Schwarzschild black hole. In previous studies, the solution to the d-dimensional cosmic string equation coupled with Poschl-Teller and Manning-Rosen potentials has not been studied. In this work, it was applied to calculate the thermodynamic properties of the Schwarzschild black hole. Renyi entropy has been determined using the wave function generated by the system. This quantity can be used in many research areas, such as measuring genetic diversity, quantifying neural activity, and network anomaly detection. Meanwhile, the thermodynamic properties of the Schwarzschild black hole that we are concerned about included the temperature and the heat capacity of the black hole. The Schwarzschild black hole is used to describe the relationships among thermal equilibria but not the transitions between equilibria. The thermodynamic study on the Schwarzschild black hole is very helpful in understanding de Sitter spacetimes because other analyzes of perturbations under the Schwarzschild black hole background are more restrictive than thermodynamic analysis in de Sitter spacetimes. So, it is important to explore more about the thermodynamics quantity of this system. The limitation of this work is that we look for the Renyi entropy without other types of entropy, such as Shannon entropy or Tsallis entropy. Likewise, the thermodynamic quantities determined in this work are only limited to the average temperature and heat capacity, not including the average vibrational energy, vibrational entropy, average free energy of vibration, and other system properties. The study of the properties of the Schwarzschild black holes can be a development material for further research.

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