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The Graceful Labeling on *W* Graph

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Abstract: Let G be a graph with vertex set *V* = *V (G)* and edge set *E* = *E(G)*. Graceful labeling is an injective function g from the vertex set *V* to a set of number {0,1,2,…,|*E*|} which induces a bijective function g^' from the set E to the set of number {0,1,2,…,|*E*|}, where for each edge *uv* ∈ *E* with *u*,*v* ∈ *V* applies *g'*(*uv*) = |g(*u*)-g(*v*)|. A graph with graceful labeling is called a graceful graph. This research aims to construct a new graph, namely a *W* graph, and prove that the *W* graph is graceful. *W* graph is a graph constructed from two ladder graphs and one C_3 graph, where C_3 is formed by connecting the end vertices of each ladder, for example, v_1 and x_1, and by adding a vertex connected to the vertices v_1 and x_1. In this paper, the authors show that the *W* graph satisfies the graceful labeling so that the *W* graph is graceful.

Keywords: graceful labeling, graph labeling, *W* graph.

宽图上的优雅标注

摘要: 設 G 是頂點集*伏*=*伏(G)* 和邊集*乙*=*乙(G)* 的圖。 優雅標記是從頂點集*伏*到一組數 {0,1,2,…,| 乙|} 的單射函數 g，它從集合乙到數集 {0, 1,2,…,| 乙|}，其中對於每條邊*紫外线*∈乙和*紫外线*∈*伏*應用 *g*'(*紫外线*)= |*克（你）-克（五）*|。帶有優雅標記的圖稱為優雅圖。本研究旨在構建一個新的圖，即*宽*圖，並證明*宽*圖是優雅的。*宽*圖是由兩個梯形圖和一個 C_3 圖構成的圖，其中 C_3 是通過連接每個梯形的末端頂點，例如 v_1 和 x_1，並通過添加連接到頂點 v_1 和 x_1 的頂點形成的。 在本文中，我們證明了宽圖滿足優雅標記，因此宽圖是優雅的。

关键词：优美标注，图标注，*宽*图。

1. Introduction

A graph *G* consists of a finite set of vertices *V(G)* and a set of edges *E(G)* consisting of distinct, unordered pairs of vertices [1]. The graph discussed in this paper is a simple, undirected, and finite graph. | E | represents the number of vertices on graph *G*, and the number of edges on graph *G* is represented by |*V*|. Graph labeling has been studied since the 60s. Graph labeling is a branch of graph theory that continues to develop. Labeling on a graph is the assignment of an integer value to the elements of the graph, usually a positive integer. Alex Rosa first discovered graceful labeling in 1967 [2]. Since this discovery, many researchers have been interested in looking for graceful labeling constructs and their variations. Graceful labeling is an injective function g from the set of vertices V to the set of numbers {0,1,2,...,|*E*|} which induces a discrete function g' from the edge set E to the set numbers {1,2,...,|*E*|} , where each edge *uv* ∈ *E* with vertex *u*,*v* ∈ *V* applies *g*'(*uv*)=|*g*(*u*)-*g*(*v*)|. Several graphs with graceful labeling include tree graphs with vertices less or equal to 35, circle graph C_n for $n = 0(mod 4)$ or $n = 3(mod 4)$, and wheel graph W_n. Another class of

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graphs known to have graceful labeling can be seen in the survey conducted by Gallian [3].

The following shows some relevant research: graceful labeling of paths [4], graceful labeling of pendant graphs [5, 6, 7], vertex graceful labeling of caterpillar graphs [8], graceful labeling on torch graph [9], counting graceful labelings of trees [10], and other results on super graceful labeling of graphs [11]. Motivated by the above articles, the authors propose the construction of graceful on *W* graphs in this paper. Furthermore, it has been proven that the W graph satisfies the graceful labeling so that the graph W is a graceful graph.

The organization of the paper is as follows. The authors present the construction of the *W* graph in the second section. In the third section, the authors prove that the W graph is graceful. Finally, the authors concluded the paper in section four.

The research method used is a literature study by studying journal articles and books related to the research topic. Furthermore, the results of this literature study are used as a theoretical basis for obtaining graceful labeling on the *W* graph.

2. Preliminaries

Previously, the authors give some definitions for constructing a new graph in this paper.

Definition 2.1. kC_4-snake graphs with $k \geq 1$ is a connected graph with k blocks whose block-cut point graph is a path, and each of the k blocks is isomorphic to C_4.

Definition 2.2. Ladder graph L_n or $P_n \times P_2$ is defined as the Cartesian product of path graph P_n and path graph P_2.

The authors construct a new graph from these definitions, which is called as W graph.

Definition 2.3. W graph, denoted by W^n, is a graph constructed by two ladder graphs $L(n+1), n \geq 1$, and a circle graph C_3 where a circle graph C_3 is formed by connecting the vertices of the ends of two ladders, for example, v_1 and x_1, and add a vertex connected to vertices, v_1 and x_1.

The following Figure 1 shows the graph W^1.

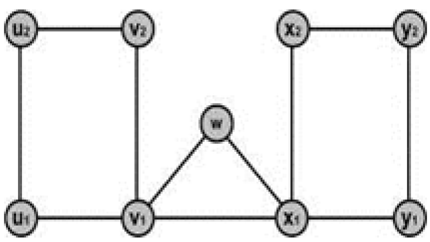

Fig. 1 The graph W^1

The graph W^n, with its vertices, is shown in Figure 2.

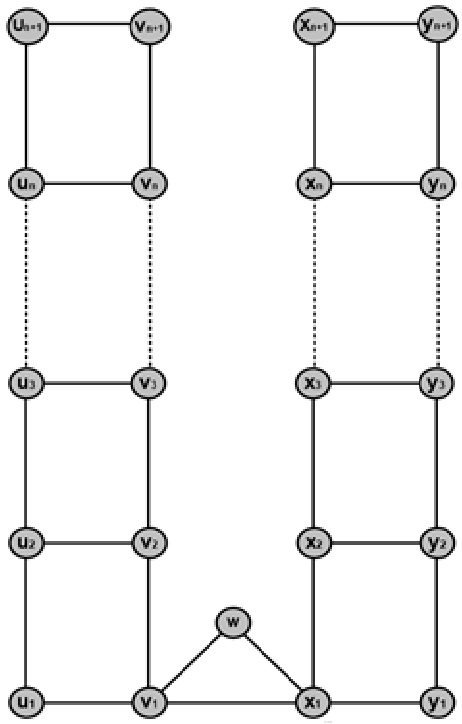

Fig. 2 The graph W^n

3. Main Result

In this section, the authors prove that the W graph is graceful.

Theorem 3.1. W graph is a graceful graph.

Proof. The graph W^n has $|V| = 4n + 5$ and $|E| = 6n + 5$. The set of vertices and edges of the graph W^n, respectively, are

$$V(W^n) = \{u_i | 1 \leq i \leq n+1\} \cup \{v_i | 1 \leq i \leq n+1\} \cup \{w\} \cup \{x_i | 1 \leq i \leq n+1\} \cup \{y_i | 1 \leq i \leq n+1\}$$

and

$$E(W^n) = \{u_i u_{i+1} | 1 \leq i \leq n\} \cup \{v_i v_{i+1} | 1 \leq i \leq n\} \cup \{u_i v_i | 1 \leq i \leq n+1\} \cup \{v_1 w\} \cup \{x_1 w\} \cup \{v_1 x_1\} \cup \{x_i x_{i+1} | 1 \leq i \leq n\} \cup \{y_i y_{i+1} | 1 \leq i \leq n\} \cup \{x_i y_i | 1 \leq i \leq n+1\}$$

There exists injective function g from $V(W^n)$ to $\{0,1,2,\ldots,6n+5\}$. Consider a labeling of vertices $g: V(W^n) \rightarrow \{0,1,2,\ldots,6n+5\}$ on graph W^n defined as follows:

$$g(w) = 0 \tag{1}$$

$$g(u_1) = 1 \tag{2}$$

$g(u_i) = 3n + \frac{3}{2} + \frac{3}{2}i,$ if i is odd, and

$$\begin{cases} 3 \leq i \leq n, n \text{ is odd} \\ 3 \leq i \leq n+1, n \text{ is even} \end{cases} \tag{3}$$

$g(u_i) = 3n + 5 - \frac{3}{2}i,$ i is even, and

$$\begin{cases} 2 \leq i \leq n+1, n \text{ is odd} \\ 2 \leq i \leq n, \ n \text{ is even} \end{cases} \tag{4}$$

$$g(v_1) = 6n + 4 \tag{5}$$

$g(v_i) = 3n + \frac{11}{2} - \frac{3}{2}i$, i is odd, and $\begin{cases} 3 \le i \le n, n \text{ is odd} \\ 3 \le i \le n+1, n \text{ is even} \end{cases}$ (6)

$g(v_i) = 3n + 1 + \frac{3}{2}i$, i is even, and $\begin{cases} 2 \le i \le n+1, n \text{ is odd} \\ 2 \le i \le n, \ n \text{ is even} \end{cases}$ (7)

$g(x_i) = 6n + \frac{13}{2} - \frac{3}{2}i$, i is odd, and $\begin{cases} 1 \le i \le n, n \text{ is odd} \\ 1 \le i \le n+1, \ n \text{ is even} \end{cases}$ (8)

$g(xi) = 1 + \frac{3}{2}i$, i is even, and $\begin{cases} 2 \le i \le n+1, n \text{ is odd} \\ 2 \le i \le n, \ n \text{ is even} \end{cases}$ (9)

$g(y_i) = \frac{3}{2} + \frac{3}{2}i$, i is odd, and $\begin{cases} 1 \le i \le n, \ n \text{ is odd} \\ 1 \le i \le n+1, \ n \text{ is even} \end{cases}$ (10)

$g(y_i) = 6n + 6 - \frac{3}{2}i$, i is even, and $\begin{cases} 2 \le i \le n+1, n \text{ is odd} \\ 2 \le i \le n, n \text{ is even} \end{cases}$. (11)

Case 1: if n is odd. From equations (1)-(11), the authors get the set of vertex labels $g(V(W^n))$ as

$\{g(w)\} \cup \{g(u_1)\} \cup \{g(u_i)| i \text{ is odd}, 3 \le i \le n\} \cup \{g(u_i)| i \text{ is even}, \ 2 \le i \le n+1\} \cup \{g(v_1)\} \cup \{g(v_i)| i \text{ is odd}, 3 \le i \le n\} \cup \{g(v_i)| i \text{ is even}, 2 \le i \le n+1\} \cup \{g(x_i)| i \text{ is odd}, 1 \le i \le n\} \cup \{g(x_i)| i \text{ is even}, 2 \le i \le n+1\} \cup \{g(y_i)| i \text{ is odd}, 1 \le i \le n\} \cup \{g(y_i)| i \text{ is even}, 2 \le i \le n+1\}$

Thus,

$$\begin{aligned}
g(V(W^n)) &= \{0\} \cup \{1\} \cup \\
&\left\{3k \middle| n+2 \le k \le \tfrac{3}{2}n + \tfrac{1}{2}\right\} \cup \\
&\left\{3k+2 \middle| \tfrac{1}{2}n + \tfrac{1}{2} \le k \le n\right\} \cup \{6n+4\} \cup \\
&\left\{3k+1 \middle| \tfrac{1}{2}n + \tfrac{3}{2} \le k \le n\right\} \cup \\
&\left\{3k+1 \middle| n+1 \le k \le \tfrac{3}{2}n + \tfrac{1}{2}\right\} \cup \\
&\left\{3k+2 \middle| \tfrac{3}{2}n + \tfrac{3}{2} \le k \le 2n+1\right\} \cup \\
&\left\{3k+1 \middle| 1 \le k \le \tfrac{n}{2} + \tfrac{1}{2}\right\} \cup \left\{3k \middle| 1 \le k \le \tfrac{n}{2} + \tfrac{1}{2}\right\} \cup \\
&\left\{3k \middle| \tfrac{3}{2}n + \tfrac{3}{2} \le k \le 2n+1\right\} = \left\{3k \middle| 1 \le k \le \tfrac{n}{2} + \tfrac{1}{2}\right\} \cup \\
&\left\{3k \middle| n+2 \le k \le \tfrac{3}{2}n + \tfrac{1}{2}\right\} \cup \\
&\left\{3k \middle| \tfrac{3}{2}n + \tfrac{3}{2} \le k \le 2n+1\right\} \cup \\
&\left\{3k+1 \middle| 1 \le k \le \tfrac{1}{2}n + \tfrac{1}{2}\right\} \cup \\
&\left\{3k+1 \middle| \tfrac{1}{2}n + \tfrac{3}{2} \le k \le n\right\} \cup \\
&\left\{3k+1 \middle| n+1 \le k \le \tfrac{3}{2}n + \tfrac{1}{2}\right\} \cup \\
&\left\{3k+2 \middle| \tfrac{1}{2}n + \tfrac{1}{2} \le k \le n\right\} \cup \\
&\left\{3k+2 \middle| \tfrac{3}{2}n + \tfrac{3}{2} \le k \le 2n+1\right\}.
\end{aligned}$$

From the formula of vertex labels (1)-(11) of the graph W^n, the authors can see that labels of all vertices are all different from each other and are a subset of $\{0,1,2,\dots,6n+5\}$. Therefore, $g: V(W^n) \to \{0,1,2,\dots,6n+5\}$ is injective.

Then, g mapping induces a g^* mapping of $E(W^n)$ to the set $\{1,2,3,\dots,6n+5\}$ defined by $g^*(uv) = |g(u) - g(v)|$, $\forall uv \in E(W^n)$, and it is a bijective function.

By using the vertex label, the authors can have label of edges $g^*: V(W^n) \to \{1,2,3,\dots,6n+5\}$ as follows:

$g^*(wx_1) = 6n+5$ (12)

$g^*(wv_1) = 6n+4$ (13)

$g^*(u_1v_1) = 6n+3$ (14)

$g^*(u_1u_2) = 3n+1$ (15)

$g^*(u_iu_{i+1}) = 3i - 2$, and $\begin{cases} 3 \le i \le n, i \text{ is odd} \\ 2 \le i \le n-1, i \text{ is even} \end{cases}$ (16)

$g^*(v_1v_2) = 3n$ (17)

$g^*(v_iv_{i+1}) = 3i - 3$, and $\begin{cases} 3 \le i \le n, i \text{ is odd} \\ 2 \le i \le n-1, i \text{ is even} \end{cases}$ (18)

$g^*(x_ix_{i+1}) = 6n + 4 - 3i$, and $\begin{cases} 1 \le i \le n, i \text{ is odd} \\ 2 \le i \le n-1, i \text{ is even} \end{cases}$ (19)

$g^*(y_iy_{i+1}) = 6n + 3 - 3i$, and $\begin{cases} 1 \le i \le n, i \text{ is odd} \\ 2 \le i \le n-1, i \text{ is even} \end{cases}$ (20)

$g^*(v_1x_1) = 1$ (21)

$g^*(u_iv_i) = 3i - 4$, and $\begin{cases} 3 \le i \le n, i \text{ is odd} \\ 2 \le i \le n+1, i \text{ is even} \end{cases}$ (22)

$g^*(x_iy_i) = 6n + 5 - 3i$, and $\begin{cases} 1 \le i \le n, i \text{ is odd} \\ 2 \le i \le n+1, i \text{ is even} \end{cases}$ (23)

From the edge labels (12)-(23), the authors can see that labels of all edges are all different and surjective. Those take all members of $\{1,2,3,\dots,6n+5\}$. Thus, $g^*: E(W^n) \to \{1,2,3 \dots,6n+5\}$ is bijective since the vertex labeling function g is injective and induces a bijective labeling function g^*, so g is graceful labeling on the W graph.

Case 2: if n is even. From equations (1)-(11), w the authors e get the set of vertex labels $g(V(W^n))$ as

$\{f(w)\} \cup \{g(u_1)\} \cup \{g(u_i) | i \text{ odd}, 3 \le i \le n+1\} \cup \{g(u_i) | i \text{ is even}, 2 \le i \le n\} \cup \{g(v_1)\} \cup \{g(v_i) | i \text{ is odd}, 3 \le i \le n+1\} \cup \{g(v_i) | i \text{ is even}, 2 \le i \le n\} \cup \{g(x_i) | i \text{ is odd}, 1 \le i \le n+1\} \cup \{g(x_i) | i \text{ is even}, 2 \le i \le n\} \cup \{g(y_i) | i \text{ is odd}, 1 \le i \le n+1\} \cup \{g(y_i) | i \text{ is even}, 2 \le i \le n\}$

Thus,

$$f(V(W^n)) = \{0\} \cup \{1\} \cup \left\{3k \middle| n+2 \le k \le \tfrac{3}{2}n+1\right\} \cup \left\{3k+2 \middle| \tfrac{1}{2}n+1 \le k \le n-2\right\} \cup \{6n+4\} \cup \left\{3k+1 \middle| \tfrac{1}{2}n+1 \le k \le n\right\} \cup \left\{3k+1 \middle| n+1 \le k \le \tfrac{3}{2}n\right\} \cup \left\{3k+2 \middle| \tfrac{3}{2}n+1 \le k \le 2n+1\right\} \cup \left\{3k+1 \middle| 1 \le k \le \tfrac{n}{2}\right\} \cup \left\{3k \middle| 1 \le k \le \tfrac{n}{2}+1\right\} \cup \left\{3k \middle| \tfrac{3}{2}n+2 \le k \le 2n+1\right\} = \left\{3k \middle| 1 \le k \le \tfrac{n}{2}+1\right\} \cup \left\{3k \middle| n+2 \le k \le \tfrac{3}{2}n+1\right\} \cup \left\{3k \middle| \tfrac{3}{2}n+2 \le k \le 2n+1\right\} \cup \left\{3k+1 \middle| 1 \le k \le \tfrac{n}{2}\right\} \cup \left\{3k+1 \middle| \tfrac{1}{2}n+1 \le k \le n\right\} \cup \left\{3k+1 \middle| n+1 \le k \le \tfrac{3}{2}n\right\} \cup \left\{3k+2 \middle| \tfrac{1}{2}n+1 \le k \le n-2\right\} \cup \left\{3k \middle| \tfrac{3}{2}n+2 \le k \le 2n+1\right\}.$$

From the vertex labels, the authors can see that labels of all vertices are different, then g is an injective function. Next, the induced edge labeling $g^*$is defined by $g^*(uv) = |g(u) - g(v)|$.

By using the vertex label, label of edges $g^*: V(W^n) \to \{1, 2, 3, \dots, 6n+5\}$ is as follows:

$$g^*(wx_1) = 6n+5 \quad (24)$$

$$g^*(wv_1) = 6n+4 \quad (25)$$

$$g^*(u_1v_1) = 6n+3 \quad (26)$$

$$g^*(u_1u_2) = 3n+1 \quad (27)$$

$$g^*(u_iu_{i+1}) = 3i - 26, \text{ and } \begin{cases} 3 \le i \le n-1, i \text{ is odd} \\ 2 \le i \le n, i \text{ is even} \end{cases} \quad (28)$$

$$g^*(v_1v_2) = 3n \quad (29)$$

$$g^*(v_iv_{i+1}) = 3i - 3, \text{ and } \begin{cases} 3 \le i \le n-1, i \text{ is odd} \\ 2 \le i \le n, i \text{ is even} \end{cases} \quad (30)$$

$$g^*(x_ix_{i+1}) = 6n + 4 - 3i, \text{ and } \begin{cases} 1 \le i \le n-1, i \text{ is odd} \\ 2 \le i \le n, i \text{ is even} \end{cases} \quad (31)$$

$$g^*(y_iy_{i+1}) = 6n + 3 - 3i, \text{ and } \begin{cases} 1 \le i \le n-1, i \text{ is odd} \\ 2 \le i \le n, i \text{ is even} \end{cases} \quad (32)$$

$$g^*(v_1x_1) = 1 \quad (33)$$

$$g^*(u_iv_i) = 3i - 4, \text{ and } \begin{cases} 3 \le i \le n-1, i \text{ is odd} \\ 2 \le i \le n, i \text{ is even} \end{cases} \quad (34)$$

$$g^*(x_iy_i) = 6n + 5 - 3i, \text{ and } \begin{cases} 1 \le i \le n+1, i \text{ is odd} \\ 2 \le i \le n, i \text{ is even} \end{cases} \quad (35)$$

From the edge labels (24)-(35), the authors can see that labels of all edges are all different and surjective then g^* is a bijective function. Since g is injective and g^* is bijective, the authors conclude that the W graph is graceful.

Figure 3 shows an example of graceful labeling for W^1.

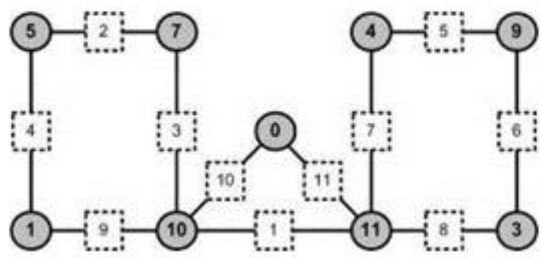

Fig. 3 Graceful labeling for W[1]

Figure 4 shows an example of graceful labeling for W^2.

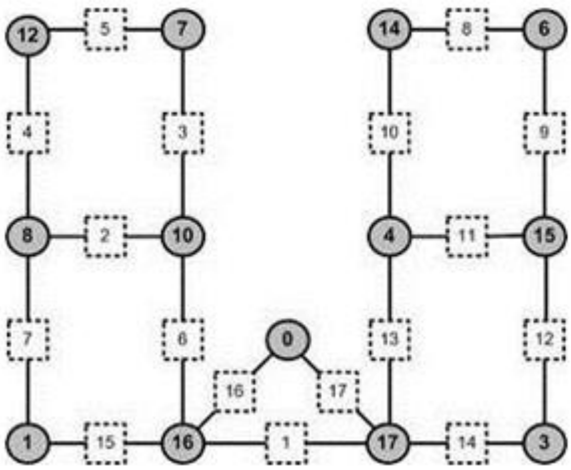

Fig. 4 Graceful labeling for W²

Figure 5 shows an example of graceful labeling for W^4.

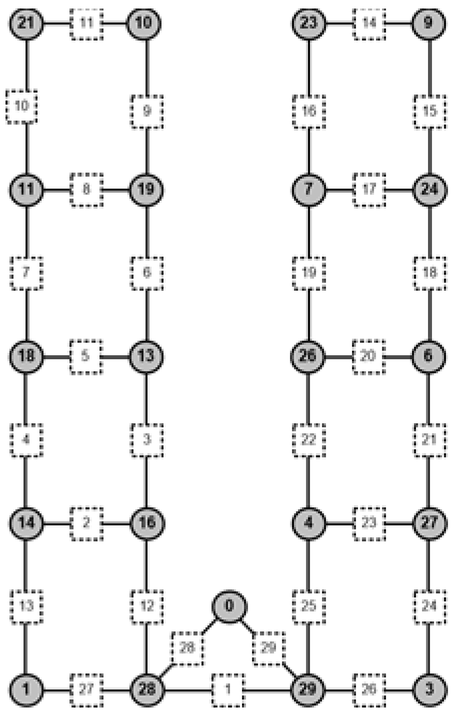

Fig. 5 Graceful labeling for W^4

4. Conclusion

Graph labeling is a topic of combinatoric mathematics research that has developed very rapidly in recent years. In this article, the authors have constructed a new graph, namely a W graph, denoted by W^n. The W graph is formed from two ladder graphs and a circle graph. More precisely, W graph is a graph constructed by two ladder graphs $L(n+1), n \geq 1$, and a circle graph C_3 where a circle graph C_3 is formed by connecting the vertices of the ends of two ladders, for example, v_1 and x_1, and add a vertex connected to vertices, v_1 and x_1. Other authors have never discussed the discussion of this graph and its labeling. The authors have defined a labeling vertex g of the vertex set V of the W graph to a set of numbers $\{0, 1, 2, \ldots, |E|\}$, with and $|E| = 6n + 5$. The labeling of the vertices g is a piecewise function, as described in the discussion. It can be shown that labeling the vertex g induces a bijective function g^* from the set E to the set of numbers $\{0,1,2, \ldots, |E|\}$, where for each edge $uv \in E$ with $u, v \in V$ applies $g^*(uv) = |g(u) - g(v)|$. The result showed that the vertex labeling function g is injective and induces a bijective labeling function g^*. Therefore, the authors have proved that the W graph is graceful.

This research only discusses graceful labeling on the newly constructed graph. There are still many types of graphs that are not known, graceful or not. This can be used for future research.

Acknowledgments

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