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# Periodic Solutions for Different Classes of Abel's Type Equation

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**Abstract:** The maximum number of periodic solutions for the ordinary differential equation is computed in this article. Periodic orbits for first order non-autonomous differential equation of Able's type from a fine focus z=0 are investigated. The research goal of the article is to achieve the maximum possible periodic solutions for two distinct polynomials; algebraic and trigonometric coefficients. The scientific novelty of the article lies in the fact that we have used our newly developed formula ϰ_10. With this formula, we determined the highest known multiplicity 10 for the classes C10,1, C10,2 with algebraic and C8,8, C20,10 with trigonometric coefficients, which is never done before in literature. We used the computer algebra program Maple 18 to handle the complicated and time-consuming integral computations required for calculating the periodic solutions. We used the systematic bifurcation method, which occurred when the coefficients are perturbed. The stability of limit cycles belonging to the classes, as mentioned earlier, is also briefly discussed. For the implementation of theoretical concepts, numerous examples are presented. Consequently, the results presented here are original, authentic, and a valuable contribution to the literature.

**Keywords:** Abel's equation, periodic solutions, perturbation method, homogeneous and non-homogeneous trigonometric coefficients.

# 不同類別的亞伯型方程的周期解

**摘要:** 在本文中，计算了常微分方程周期解的最大数目。研究了来自细焦点 z=0 的有能力的型一阶非自治微分方程的周期轨道。文章的研究目标是实现两个不同多项式的最大可能周期解；代数和三角系数。这篇文章的科学新颖之处在于我们使用了我们新开发的公式 ϰ_10。使用此公式，我们确定了代数类 C10,1、C10,2 和三角系数类 C8,8、C20,10 的最高已知多重性 10，这在文献中从未做过。我们使用计算机代数程序 楓 18 来处理计算周期解所需的复杂且耗时的积分计算。我们使用了系统分叉方法，它发生在系数被扰动时。还简要讨论了属于上述类别的极限环的稳定性。对于理论概念的实施，提供了许多示例。因此，这里呈现的结果是原创的、真实的，并且对文献做出了宝贵的贡献

**关键词：**阿贝尔方程, 周期解, 摄动方法, 齐次和非齐次三角系数。

## 1. Introduction

Limit cycles have been used to model the behavior of many real-world oscillatory systems. Many scientific applications depend on these, including aerodynamic limit cycle oscillations, cancer cell migration in confined microenvironments, electrical equipment, and engineering. Non-autonomous ODE's model is used in chemical engineering and physical metallurgy to deal with single-phase reaction kinetics in a well-stirred reaction vessel. The analysis of periodic solutions of nonlinear non-autonomous differential equations is becoming increasingly important due to this.

A limit cycle is a closed trajectory with the property that at least one other trajectory spirals through as time reaches infinity while studying a dynamical system with a two-dimensional phase space. The perturbation of the periodic orbits of the center is a classic method

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for obtaining the limit cycles. Seasonal factors, which control external effects, are therefore considered non-autonomous models. Non-autonomous periodic models can also be studied from [9,10,15].

The non-autonomous differential equation for which limit cycles are calculated by perturbing the periodic solutions of the center is also the focus of this paper. Limit cycles, or discrete periodic trajectories of polynomial vector fields, are challenging to study. Although there are many tools to establish such cycles in a specific domain, these apply to only a particular differential equation. David Hilbert [7] uses equation (6) to question the maximum number of limit cycles. Although it is still an open question, he restricts the coefficients (if possible) to calculate the maximum number of limit cycles using some perturbation techniques.

Finding limit cycles is a challenging problem in general. We are considering the differential equation of the form:

$$\dot{z} = A(u)z^3 + B(u)z^2 + C(u)z \tag{1}$$

where independent variable 'u' and coefficients $A, B, C \in \mathbb{R}$ but $z \in \mathbb{C}$. The general problem is obtaining the maximum number of periodic solutions of (1), which continues to attract more interest. Equation (1) is the part of the equation that is defined as follows in Lloyd [8]:

$$\dot{z} = \mathrm{f}_0(\mathrm{u})\mathrm{z}^{\mathrm{n}} + \mathrm{f}_1(\mathrm{u})\mathrm{z}^{\mathrm{n}-1} + \mathrm{f}_2(\mathrm{u})\mathrm{z}^{\mathrm{n}-2} + \ldots + \mathrm{f}_n(\mathrm{u}) \tag{2}$$

With the assumption that $\mathrm{f}_0(u) = 1$ and $\mathrm{f}_1(u), \ldots, \mathrm{f}_n(u)$ are real-valued, continuous, periodic functions. This equation cannot be solved in this general form. We will discuss here the situations where periodic solutions of this equation exist. For $n = 3$, the differential equation (2) is known as Abel's differential equation, and it is important due to its similarity with Hilbert's sixteenth problem for differential equations. For degree $n \geq 4$, Abel's equation may have an arbitrary number of limit cycles. The results of [8] no longer hold; indeed, Neto [12] has provided examples that show that there is no upper bound on the periodic solutions of (1). Also, assume that for equation (1), there exists $p \in \mathbb{R}$ such that:

$$z(p) = z(0) \tag{3}$$

The form (3) solutions are periodic, even if $A, B$, and $C$ of equation (1) are not periodic. For $z = 0$, how to compute multiplicity "μ" is explained in [1]; for the sake of ease, we explained it here briefly. We write $z(u, 0, r) = \sum_{i=1}^{\infty} q_i(u) r^i$ for $0 \leq u \leq p$ also $r$ lies in the regions nearby $z = 0$.

Alwash in [2], concerned mainly with equation (1) when the multiplicity μ, is $\mu > 1$, and $C(u) = 0$ without loss of generality. As a result, equation (1) takes the form as:

$$\dot{z} = A(u)z^3 + B(u)z^2 \tag{4}$$

Here A and B are polynomials (i) in u (ii) in $cos(u)$ and $\sin(u)$, for more details, see [1,3-6]. Complexified form is used here. When the coefficients of A and B are perturbed slightly, limit cycles bifurcate out of the fine focus $z = 0$; the limit cycles thus produced are said to be of small amplitude. The functions $\dot{q}_i(u)$, for $i > 1$ are computed with the help of the following equation:

$$\dot{q}_i = A(z)\left(\sum_{\substack{j+k+l=i \\ j,k,l \geq 1}}\right)(q_j q_k q_l) + B(z)\left(\sum_{\substack{j+k=i \\ j,k \geq 1}}\right)(q_j q_k) \tag{5}$$

With $q_1(z) = 1$. For $i \leq 8$ and $i = 9$, functions $q_i(z)$ and $\varkappa_i$ are provided in [2] and [13] accordingly, and for $i = 10$, we have calculated formulae presented in Theorem 2.1.

This article is organized as follows: In Section 2 and 3, some essential formulas by which we calculate the multiplicity of the periodic solution, and then we have described the conditions under which (1) have a center. Section 4 presents the main findings; the coefficients are trigonometric and algebraic for equation (1). In section 5, some examples are presented to validate the result given in section 4. In the last section of this article, the conclusions are discussed.

## 2. Development of Essential Formulas

For equation (5), certain functions with our newly developed formulae are presented in theorems 2.1. For example, for the calculation of $\varkappa_{10}$, we used the relation (5) with $i = 10$.

*Theorem 2.1:* The solution $z = 0$ of (5) has a multiplicity k, wherever $2 \leq k \leq 10$ if and only if $\varkappa_n = 0$ for $2 \leq n \leq k-1$ and $\varkappa_n \neq 0$ where:

$$\varkappa_2 = \int_0^p B,$$

$$\varkappa_3 = \int_0^p A,$$

$$\varkappa_4 = \int_0^p A\bar{B},$$

$$\varkappa_5 = \int_0^p A\bar{B}^2,$$

$$\varkappa_6 = \int_0^p \left(A\bar{B}^3 - \tfrac{1}{2}\bar{A}^2 B\right),$$

$$\varkappa_7 = \int_0^p \left(A\bar{B}^4 + 2A\bar{B}^2\bar{A}\right),$$

$$\varkappa_8 = \int_0^p \left(A\bar{B}^5 + 3A\bar{B}^3\bar{A} + A\bar{B}^2\overline{\bar{B}A} - \tfrac{1}{2}\bar{A}^3 B\right),$$

$$\varkappa_9 = \int_0^p \left(A\bar{B}^6 - 5A\bar{B}^4\bar{A} - 2\bar{B}^3\overline{B\bar{A}} + 20\overline{B\bar{A}^2} + 2\overline{B\bar{A}}B\bar{A}^2\right),$$

and

$$\begin{aligned}\varkappa_{10} = \int_0^p \Big(A\bar{B}^7 &- \frac{1235}{6}\bar{A}\bar{B}^5 - \frac{970}{3}A\bar{A}^2\bar{B}^3 \\ &- 237B\bar{B}^2\bar{A}^3 - 24A\bar{A}^2B\bar{B}^2 - 70\bar{A}\bar{B}^3A^2 \\ &- 74A\bar{A}^3\bar{B} + \frac{5}{2}\bar{A}^2B\bar{B}^4 + 32\bar{B}^4A\overline{B\bar{A}} \\ &- 36B\bar{B}\bar{A}^2\overline{B\bar{A}} - 16B\bar{B}^4\bar{A} - 15\bar{B}^5A^2 \\ &- 21\bar{A}^4B - 8B\bar{B}^4A\bar{A}\Big).\end{aligned}$$

*Proof:* By definition, the multiplicity of the zero solution is k if $a_n(\mathrm{p}) = 0$ for $2 \leq n \leq k-1$ and $u_k(\mathrm{p}) \neq 0$. Write $\xi_k = u_k(\mathrm{p})$ and let $\varkappa_k$ be the value of $\xi_k$, when $\xi_n = 0$ for $n < k$.

Since $u_2(z) = \int_0^p B$, we have $\xi_2 = \int_0^p B$. With $\xi_2 = 0$, we have $\xi_3 = \int_0^p A$ and $\xi_4 = \int_0^p A\bar{B}$; hence $\varkappa_3$ and

$\varkappa_4$ as stated. Next, suppose that $\xi_2, \xi_3 = 0$ , and substitute the relations $\int_o^p B = \int_o^p A = 0$ into the expressions for $u_5$(p) and $u_6$(p) given in ([5], theorem 2.1); we obtain $\varkappa_4 = \int_o^p A\bar{B}$, $\varkappa_5 = \int_o^p A\bar{B}^2$. Now, we suppose that $\xi_5 = 0$ (as well as $\xi_2 = \xi_3 = \xi_4 = 0$). We substitute the relations $\int_o^p B$, $\int_o^p A$, $\int_o^p A\bar{B}$, $\int_o^p A\bar{B}^2 = 0$ into the expressions for $u_7$(σ) and $u_8$(σ); then $\varkappa_7 = \int_o^p (A\bar{B}^4 + 2A\bar{B}^2\bar{A})$, and $\varkappa_8 = \int_o^p (A\bar{B}^5 + 3A\bar{B}^3\bar{A} + A\bar{B}^2\overline{\bar{B}A} - \frac{1}{2}\bar{A}^3 B)$. Continuing in the same way, we get $\varkappa_9 = \int_o^p (A\bar{B}^6 - 5A\bar{B}^4\bar{A} - 2\bar{B}^3\overline{B\bar{A}} + 20\overline{B\bar{A}^2} + 2\overline{B\bar{A}}B\bar{A}^2)$,

And $\varkappa_{10} = \int_o^p (A\bar{B}^7 - \frac{1235}{6}\bar{A}\bar{B}^5 - \frac{970}{3}A\bar{A}^2\bar{B}^3 - 237B\bar{B}^2\bar{A}^3 - 24A\bar{A}^2B\bar{B}^2 - 70\bar{A}\bar{B}^3A^2 - 74A\bar{A}^3\bar{B} + \frac{5}{2}\bar{A}^2B\bar{B}^4 + 32\bar{B}^4A\overline{B\bar{A}} - 36B\bar{B}\bar{A}^2\overline{B\bar{A}} - 16B\bar{B}^4\bar{A} - 15\bar{B}^5A^2 - 21\bar{A}^4B - 8B\bar{B}^4A\bar{A})$.

By using this theorem 2.1, we calculate the results. For this, some required conditions for the center are given in the next section. After this, the method of bifurcation is applied.

Now we are going to define the center and some necessary conditions for the center.

*Definition 2.1:* An equilibrium point surrounded in its immediate neighborhood (not necessarily over the whole plane) by a closed path is called a center.

### 2.1. Conditions for Center

The results in this section are from [1]. During the calculation of maximum multiplicity $\varkappa_k$, some good conditions for $z = 0$ as a center are now given in corollaries and theorem 2.2 as:

*Corollary 2.1:* The origin is a center for the equation (4). If A is a constant multiple of B and $\int_o^p B(u)du = 0$.

*Corollary 2.2:* If any $A(z)$ or $B(z)$ is identically zero and the other has zero mean value, the origin is a center.

*Theorem 2.2:* Suppose, there are continuous functions $f, g$ defined on $I = \sigma([0, \alpha])$ and differentiable function σ with $\sigma(\alpha) = \sigma(0)$ such that:

$$A(u) = f(\sigma(u))\dot{\sigma},$$
$$B(u) = g(\sigma(u))\dot{\sigma}.$$

Then the origin is a center for (4).

*Remark 2.1:* From the "exchange of stability" statement, we can deduce that if multiplicity is even, the origin is stable $\varkappa_\mu < 0$ but unstable if $\varkappa_\mu > 0$. If μ is odd, then the origin is stable on the right and unstable on the left if $\varkappa_\mu < 0$, and stable on the left but unstable on the right if $\varkappa_\mu > 0$.

## 3. Main Results

### 3.1. The Algebraic Coefficient

Throughout this section, we shall be dealing with calculating the maximum number of periodic solutions. We firstly understand how such solutions are created and destroyed, and then the primary concern is the local question of bifurcation. For calculation of the limit cycles, we use the programming language Maple18. We consider different classes and then find the highest possible value of the multiplicity, denoted by $\mu_{max}$. In addition, any result obtained is accurate. Now, consider the polynomial "u" for classes $C_{10,1}$ and $C_{10,2}$ for equation of the form (4); additional information can be found in articles [2,3-6,14].

*Theorem 3.1:* Consider class $C_{10,1}$ for the equation (4), with

$$A(u) = c + du + eu^2 + fu^3 + ku^8 + mu^{10},$$
$$B(u) = p + qu.$$

Then we conclude $\mu_{max}(C_{10,1}) \geq 10$.

*Proof:* Using theorem 2.1, we take:

$$\varkappa_2 = p + \frac{1}{2}q,$$
$$\varkappa_3 = c + \frac{1}{2}d + \frac{1}{3}e + \frac{1}{4}f + \frac{1}{9}k + \frac{1}{11}m.$$

Since multiplicity of $z = 0$ is $\mu = 2$, if $\varkappa_2 \neq 0$. Moreover, multiplicity is $\mu = 3$, if $\varkappa_2 = 0$ but $\varkappa_3 \neq 0$. If $\varkappa_2 = \varkappa_3 = 0$, then by using value of "$p$" and "$c$", $A(u)$ and $B(u)$ are as follows:

$$A(u) = d(u - \tfrac{1}{2}) + e(u^2 - \tfrac{1}{3}) + f(u^3 - \tfrac{1}{4}) + k(u^8 - \tfrac{1}{9}) + m(u^{10} - \tfrac{1}{11}) \quad (6)$$

$$B(u) = q\left(u - \tfrac{1}{2}\right) \quad (7)$$

In addition, we compute $\varkappa_4$ as given below:

$$\varkappa_4 = \frac{q(1350m + 1456k + 1287f + 858e)}{308880}.$$

If $\varkappa_4 = 0$, then either $q = 0$ or:

$$m = -\tfrac{1456}{1350}k - \tfrac{1287}{1350}f - \tfrac{858}{1350}e \quad (8)$$

If $q = 0$, then $B(u) = 0$, and for $\varkappa_3 = 0$, A(u) has a mean value of zero. Because of corollary 2.2, the origin is a center. Consider the case $q \neq 0$. Now, substituting (8) $\varkappa_5$ is:

$$\varkappa^5 = -\frac{q^2(406k + 1287f + 858e)}{27027000}$$

If $\varkappa_5 = 0$, then as we already considered $q \neq 0$ implies:

$$k = -\tfrac{1287}{406}f + \tfrac{858}{406}e \quad (9)$$

Moreover, by using (9) we take $\varkappa_6$ as:

$$\varkappa_6 = -\frac{q(2e + 3f)(511224e - 874437q^2 + 475357f)}{10580255804160}$$

If $\varkappa_6 = 0$, as already considered $q \neq 0$ either $e = -\frac{3}{2}f$ or

$$e = \tfrac{874437}{511224}q^2 - \tfrac{475357}{511224}f \quad (10)$$

If $e = -\frac{3}{2}f$, then $k = m = 0$, and the equations (6) & (7) takes the form: $A(u) = d\left(u - \frac{1}{2}\right) + f\left(u^3 - \frac{3}{2}u^2 + \frac{1}{4}\right)$, & $B(u) = q\left(u - \frac{1}{2}\right)$.

Let $\sigma(u) = \frac{u^2}{2} - \frac{u}{2}$ then, $\dot{\sigma}(u) = u - \frac{1}{2}$, also $\sigma(0) = \sigma(1)$. So, the above equations can be written as:

$$A(u) = \left[d + f\left(u^2 - u - \frac{1}{2}\right)\right]\dot{\sigma}(u), B(u) = q\dot{\sigma}(u)$$

Therefore, from theorem 2.2, origin is a center with $f(\sigma) = d + f\left(u^2 - u - \frac{1}{2}\right)$, and $g(\sigma) = q$. So, we put $e \neq -\frac{3}{2}f$. If (10) holds then we compute $\varkappa_7$ as: $\varkappa_7 = \frac{529q^2(3q^2+f)(-32611306619f+1468413933q^2+1290162820d)}{543129265986052838400}$.

If $\varkappa_7 = 0$ recalling that $q \neq 0$ then either $f = -3q^2$ or:

$$f = \frac{146846193933}{32611306619}q^2 + \frac{129016322820}{32611306619}d \quad (11)$$

If $f = -3q^2$ then,

$$A(u) = d\left(u - \frac{1}{2}\right) + q^2\left(-3u^3 + \frac{9}{2}u^2 - \frac{3}{4}\right),$$

$$B(u) = q\left(u - \frac{1}{2}\right)$$

Theorem 2.2, gives origin is the center with $f(\sigma) = d + q^2\left(-3u^2 + 3u + \frac{3}{2}\right)$ and $g(\sigma) = q$. So, let us consider $f \neq -3q^2$. By using (11), we calculate $\varkappa_8$ as:

$$\varkappa_8 = \frac{529q(18233q^2 + 9614d)\Psi}{17603225678977464268963847065852672O}.$$

where the value of Ψ is,

$$\Psi = -12775909938923669508494253539 29q^4 + 17843560800301558376540827388dq^2 + 28936797035295870095995243 80d^2.$$

Now, if $\varkappa_8 = 0$ then either $\Psi = 0$ or

$$d = -\frac{18233}{9614}q \quad (12)$$

Because $q \neq 0$. If (12) holds but $\Psi \neq 0, q \neq 0$, we compute $\varkappa_9$ as:

$$\varkappa_9 = -\frac{q^5(391409772+4526009q)}{2422379127168}.$$

If $\varkappa_9 = 0$, as $q \neq 0$ (shown above) results $q^5 \neq 0$, we get q as a constant number:

$$q = -\frac{391409772}{4526009} \quad (13)$$

If equation (12) $\neq 0, q \neq 0$, but $\Psi = 0$ holds then $e = y_i m^2$ for $i = 1,2$ with $y_1 = 133.1766309, y_2 = -116.9999242$. If (13) holds, then we calculate $\varkappa_{10}$ as:

$$\varkappa_{10} = -\frac{933514281331515930245682043545250303398641088791807100178377941742120495877385081 5232}{579389654487942946965726551297214332117906066816491844460616391273770091215}$$

This $\varkappa_{10}$ is equal to a non-zero constant number. Therefore, it is concluded that polynomial class $C_{10,1}$ has a maximum multiplicity of 10, and mathematically we can write it as $\mu_{max}(C_{10,1}) \geq 10$.

The outcome of maximum multiplicity for classes $C_{10,1}$ is even, and the sign assigned with it is negative. So, by using remark 2.1, it can be concluded that the origin is stable, shown in Figure 1.

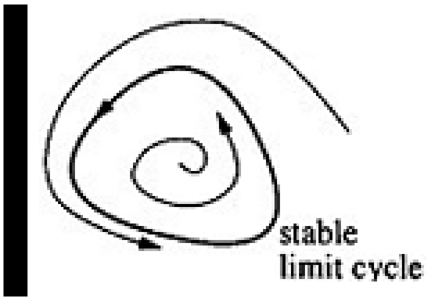


Fig. 1 Stable origin

In the following theorem, we have calculated the nontrivial real periodic solutions with the help of the perturbation method. More details of this method are presented in [1,2].

*Theorem 3.2:* For equation (4), consider $A(u), B(u)$ as given below

$$A(u) = \frac{953}{4807}\left(-\frac{391409772}{4526009} + \varepsilon_1\right)^2 - \frac{2290544372}{32611306619}\varepsilon_2 + \frac{1166225}{10735704}\varepsilon_3 - \frac{151}{609}\varepsilon_4 - \frac{97}{7425}\varepsilon_5 - \frac{1}{11}\varepsilon_6 + \varepsilon_7 + \left(-\frac{18233}{9614}\left(-\frac{391409772}{4526009} + \varepsilon_1\right)^2 + \varepsilon_2\right)u + \left(-\frac{391409772}{4526009} + \varepsilon_1\right)^2 - \frac{239929315395}{65222613238}\varepsilon_2 - \frac{1}{11}\varepsilon_6 + \varepsilon_7 + \left(-\frac{18233}{9614}\left(-\frac{391409772}{4526009} + \varepsilon_1\right)^2 + \varepsilon_2\right)u + \left(\left(-\frac{391409772}{4526009} + \varepsilon_1\right)^2 - \frac{239929315395}{65222613238}\varepsilon_2 - \frac{1437293}{1192856}\varepsilon_3 - \frac{429}{203}\varepsilon_4 + \varepsilon_5\right)u^8 + \left(\frac{241817360785}{65222613238}\varepsilon_2 + \frac{1437293}{1533672}\varepsilon_3 + \frac{143}{87}\varepsilon_4 - \frac{728}{675}\varepsilon_5 + \varepsilon_6\right)u^{10}.$$

$$B(u) = \frac{195704886}{4526009} - \frac{1}{2}\varepsilon_1 + \varepsilon_8 + \left(-\frac{391409772}{4526009} + \varepsilon_1\right)u.$$

Choose $\varepsilon_p$ to be non-zero and small as compared to $\varepsilon_{p-1}$ ; $1 \leq p \leq 8$ . After that, there are eight real nontrivial periodic solutions.

*Proof:* The proof follows the same pattern as did in [4]. So, it is omitted.

*Theorem 3.3:* Consider class $C_{10,2}$ for the equation (4), with

$$A(u) = a + bu + cu^2 + du^3 + fu^5 + ku^{10},$$
$$B(u) = l + nu^2.$$

Then we conclude $\mu_{max}(C_{10,2}) \geq 10$.

*Proof:* Using theorem 2.1, we can obtain the following:

$$\varkappa_2 = l + \frac{1}{3}n,$$

$$\varkappa_3 = a + \frac{1}{2}b + \frac{1}{3}c + \frac{1}{4}d + \frac{1}{6}f + \frac{1}{11}k.$$

Thus multiplicity of $z = 0$ is $\mu = 2$, if $\varkappa_2 \neq 0$. In addition, multiplicity is $\mu = 3$, if $\varkappa_2 = 0$ but $\varkappa_3 \neq 0$. If $\varkappa_2 = \varkappa_3 = 0$, then by using value of "$p$" and "$c$", $A(u)$ and $B(u)$ are as follows:

$$A(u) = b\left(u - \frac{1}{2}\right) + c\left(u^2 - \frac{1}{3}\right) + d\left(u^3 - \frac{1}{4}\right) + f\left(u^5 - \frac{1}{6}\right) + k\left(u^{10} - \frac{1}{11}\right) \quad (14)$$

$$B(u) = n\left(u^2 - \frac{1}{3}\right) \quad (15)$$

In addition, we compute $\varkappa_4$ as given below:

$$\varkappa_4 = -\frac{n(-600k - 550f - 297d + 462b)}{166320}.$$

If $\varkappa_4 = 0$, then either $n = 0$ or

$$k = -\frac{550}{600}f - \frac{297}{600}d + \frac{462}{600}b \qquad (16)$$

If $n = 0$ then $B(u) = 0$ and for $\varkappa_3 = 0$, the mean value of $A(u)$ is zero. Therefore, the origin is a center from corollary 2.2. Let us consider $n \neq 0$. If (16) holds, and then $\varkappa_5$ is as follows:

$$\varkappa_5 = -\frac{n(-3175f - 2046d + 1341b)}{125307000}.$$

If $\varkappa_5 = 0$ then as we already consider $n \neq 0$ implies:

$$f = -\frac{2046}{3175}d + \frac{1341}{3175}b \qquad (17)$$

Moreover, by using (17) we take $\varkappa_6$ as follows:

$$\varkappa_6 = -\frac{n(4b+d)(1860441678b - 314248893d + 2078291500n^2)}{240627002616000000}.$$

If $\varkappa_6 = 0$, then as we already consider $n \neq 0$, either $d = -4b$ or

$$d = -\frac{2078291500}{314248893}n^2 + \frac{1860441678}{314248893}b \qquad (18)$$

If $d = -4b$ then (17) gives $f = 3b$, and (16) gives $k = 0$. By using values of $d, f, k$ the equations (14) and (15) are:

$$A(u) = c(u^2 - \frac{1}{3}) + b(3u^3 - 3u)(u^2 - \frac{1}{3}),$$

$$B(u) = n(u^2 - \frac{1}{3}).$$

Let $\sigma(u) = u^3 - u$ then $\dot{\sigma}(u) = 3u^2 - 1$. Also, $\sigma(0) = \sigma(1)$. Therefore, we write the above equations as:

$$A(u) = \frac{1}{3}[c + b(3u^3 - 3u)]\dot{\sigma}, B(u) = \frac{n}{3}\dot{\sigma}.$$

Theorem 2.2 gives origin is the center with

$$f(\sigma) = \frac{1}{3}(c + b(3u^3 - 3u)), \& \ g(\sigma) = \frac{n}{3}$$

Therefore, we take $d \neq -4b$. If (l8) holds, then we compute $\varkappa_7$ as:

$$\varkappa_7 = -\frac{1423n^2(2n^2+3b)(80356837394925b + 61014434424150n^2 + 18611046510566c)}{695459987763823412466336}.$$

If $\varkappa_7 = 0$ recalling that $n \neq 0$, then either $b = -\frac{2}{3}n^2$ or

$$b = -\frac{61014434424150}{80356837394925}n^2 - \frac{18611046510566}{80356837394925}c \qquad (19)$$

If $b = -\frac{2}{3}n^2$ then by substituting $\sigma(u) = u^3 - u$ and $\dot{\sigma}(u) = 3u^2 - 1$, with the condition that $\sigma(0) = \sigma(1)$. So we write $A(u) = \frac{1}{3}[c + n^2(-2u^3 + 2u)]\dot{\sigma}$, $B(u) = \frac{n}{3}\dot{\sigma}$.

From theorem 2.2, origin is the center with $f(\sigma) = \frac{1}{3}[c + n^2(-2u^3 + 2u)]$ and $g(\sigma) = \frac{n}{3}$. So let us consider $b \neq -\frac{2}{3}n^2$. By using (19) we calculate $\varkappa_8$ as:

$$\varkappa_8 = \frac{1423n(248700n^2 + 621851c)\varpi}{3575712960967298190331277392400 0}.$$

where

$$\varpi = -85670427300896487901081646030163000n^4 - 40121848700019459628102612660961600cn^2 + 5231455084985036914546633488193893c^2.$$

Now, if $\varkappa_8 = 0$ then either $\varpi = 0$ or

$$c = -\frac{248700}{621851}n^2 \qquad (20)$$

Because $n \neq 0$. If (20) holds but $\varpi \neq 0, n \neq 0$ then $\varkappa_9$ is computed as:

$$\varkappa_9 = -\frac{n^5(9528355616355837 + 115245349334257n)}{57565787121590663916}.$$

If $\varkappa_9 = 0$, we put the value of 'n' as:

$$n = -\frac{46325757047490400}{468140924873533} \qquad (21)$$

If equation $(20) \neq 0, n \neq 0$, but $\varpi = 0$ holds then $c = v_i\frac{m^2}{3}$; here $i = 1,2$ and $v_1 = 45.009923140, v_2 = -17.20305420$. If (21) holds, then we calculate $\varkappa_{10}$ as:

$$\varkappa_{10} = -\frac{1210152397379026524183232263961207474309609776413\ 87338985453701988065393476920513555898 27686327}{8631847655799227766100267651194576977383841296202 8\ 87502119786276380233637430747572042786177070}.$$

Here $\varkappa_{10}$ is equal to a non-zero number. Therefore, it is concluded that polynomial class $C_{10,2}$ has a maximum multiplicity of 10, and mathematically we can write it as $\mu_{max}(C_{10,2}) \geq 10$.

The outcome of maximum multiplicity for class $C_{10,2}$ is even, and the sign assigned is negative. Therefore, by using remark 2.1, it can be concluded that the origin is stable, as shown in Figure 1.

*Theorem 3.4:* For equation (4), consider that

$$A(u) = C_1 + bu + cu^2 + du^3 + fu^5 + ku^{10},$$

$$B(u) = \frac{3176118538785279}{115245349334257} - \frac{1}{3}\varepsilon_1 + \varepsilon_8 + (-\frac{9528355616355837}{115245349334257} + \varepsilon_1)u^2.$$

With

$$C_1 = \frac{82900}{621851}(-\frac{9528355616355837}{115245349334257} + \varepsilon_1)^2 + \frac{51034579009}{3571414995330}\varepsilon_2 - \frac{89841105}{59856932}\varepsilon_3 - \frac{3843}{25400}\varepsilon_4 - \frac{1}{12}\varepsilon_5 - \frac{1}{11}\varepsilon_6 + \varepsilon_7,$$

$$b = -(-\frac{9528355616355837}{115245349334257} + \varepsilon_1)^2 - \frac{18611046510566}{80356837394925}\varepsilon_2 + \varepsilon_3,$$

$$c = -\frac{248700}{621851}(-\frac{9528355616355837}{115245349334257} + \varepsilon_1)^2 + \varepsilon_2,$$

$$d = (-\frac{9528355616355837}{115245349334257} + \varepsilon_1)^2 - \frac{257092781756884}{187499287254825}\varepsilon_2 + \frac{620147226}{104749631}\varepsilon_3 + \varepsilon_4,$$

$$f = -2(-\frac{9528355616355837}{115245349334257} + \varepsilon_1)^2 + \frac{419748181298}{534186003575}\varepsilon_2 - \frac{355386447}{104749631}\varepsilon_3 - \frac{2046}{3175}\varepsilon_4 + \varepsilon_5,$$

And

$$k = -\frac{3805928977873}{17307626515830}\varepsilon_2 + \frac{23401235}{24646972}\varepsilon_3 + \frac{2431}{25400}\varepsilon_4 - \frac{11}{12}\varepsilon_5 + \varepsilon_6.$$

If $\varepsilon_l, (1 \leq l \leq 8)$, are assumed to be non-zero, as well as

$$|\varepsilon_8|\text{"}\ |\varepsilon_7|\text{"}\ |\varepsilon_6|\text{"}\ldots\text{"}\ |\varepsilon_1|.$$

Then (4) has eight distinct nontrivial real periodic solutions.

### 3.2. Trigonometric Coefficients

Now, we consider equation (4), with polynomials $A(u)$ and $B(u)$ in $sin(u)$ and $\cos(u)$; for this, we suppose that $\omega = 2\pi$. Some classes with trigonometric coefficients are considered for the calculation of the periodic solutions.

*3.2.1. Non-homogeneous Polynomial for Classes $C_{18,9}$ and $C_{20,10}$*

*Theorem 3.5:* Let the class $C_{18,9}$, If the coefficients are:

$A(u) = ((c)sin^9(u)\cos(u) + (d)sin^8(u)cos^2(u) + (e)cos^7(u)sin^3(u))(cos^2(u) + sin^2(u))^4$ , $\mathrm{B(u) = (acos^6(u)sin(u) + bcos(u)sin^6(u))(cos^2(u) + sin^2(u))}$.

Then we calculate $\mu_{\max}(C_{18,9}) \geq 9$.

*Proof:* Using Theorem 2.2, it is calculated that $\varkappa_2 = 0, \varkappa_3 = \frac{7d\pi}{128}, \varkappa_4 = 0$ and by proceeding further, we calculate as:

$$\varkappa_5 = -\frac{13ab\pi(5c+3e)}{7340032}.$$

If $\varkappa_5 = 0$ then, either $a = b = 0$ or:

$$5c + 3e = 0 \tag{22}$$

If $a = b = 0$ implies $B(u) = 0$, and for $\varkappa_3 = 0$. From corollary 2.2, the origin is the centre. Substituting $c = -\frac{3}{5}e$ from equation (22), we get $\varkappa_6 = 0$ with $\varkappa_7$ as:

$$\varkappa_7 = -\frac{eba\pi(50904e + 25425a^2 - 47725b^2)}{2630667468800}.$$

For $\varkappa_7 = 0$, as $\pi ab \neq 0$, also for $e = 0$, gives $c = 0$. Also $A(u)$ is zero for the value of $e = d = c = 0$, and for $\varkappa_2 = 0$ gives that origin is a center. So, $eba\pi \neq 0$. Now, substituting:

$$e = -\frac{25425}{50904}a^2 + \frac{47725}{50904}b^2,$$

We calculate $\varkappa_8 = 0$ and $\varkappa_9$ as:

$$\varkappa_9 = -\frac{\pi(1017a^2 - 1909b^2)(340429698742 4625ba^5 + 5363300039456000a^4 - 1804033741250b^3a^3 + 17503010134208a^2b^2 + 2146550583771b^5a + 7214972536774208862 7200b^4)}{2591360623050063116556296 4148224}.$$

If $\varkappa_9 = 0$, then we put $a^2 = \frac{1909}{1017}b^2$ and calculate $_{10} = 0$.

As $\varkappa_{10} = 0$, but as the last $\varkappa_i$ should always be non-zero. Hence our conclusion is $\mu_{\max}(C_{18,9}) \geq 9$.

*Theorem 3.6:* Let the class $C_{20,10}$, If:

$$A(u) = ((a)sin^{10}(u) + (b)cos\,usin^9(u) + (c)cos^9(u)sin\,(u) + (d)cos^{10}(u))(cos^2(u) + sin^2(u))^5,$$

$$B(u) = ((k)cos^5(u)sin(u) + (m)sin^5(u)cos(u))(cos^2(u) + sin^2(u))^2.$$

Then $\mu_{\max}(C_{20,10}) \geq 10$ is presented.

*Proof:* By using theorem 2.1, it is calculated that $\varkappa_2 = 0$ and $\varkappa_3 = \frac{63\pi(a+d)}{128}$.

If $\varkappa_3 = 0$, then, as $\pi \neq 0$, we put "a" in terms of "d" and get $\varkappa_4$ as:

$$\varkappa_4 = -\frac{1065\pi d(k+m)}{16384}.$$

For $\varkappa_4 = 0$, if we consider $d\pi \neq 0$, then we put $k = -m$ and get $\varkappa_5 = 0$ with $\varkappa_6$ as:

$$\varkappa_6 = -\frac{18897md\pi(b+c)}{10485760}.$$

For $\varkappa_6 = 0$, if $m = 0$, then $k = 0$, and as a result, $B(u) = 0$, and for $\varkappa_3 = 0$ shows that the mean value of $A(u)$ is zero. Consequently, the origin is a center. Therefore, we take $m \neq 0$ and put $b = -c$, with which we calculate $\varkappa_7 = \varkappa_8 = 0$, but $\varkappa_9$ as:

$$\varkappa_9 = \frac{m\pi(8583840c^2 + 557697cm^3 - 3115860d^2)}{176664084480}.$$

If $\varkappa_9 = 0$, then as $m\pi \neq 0$, we put,

$$d^2 = \frac{858384000}{31158326160}c^2 + \frac{5576597}{31158326160}cm^3,$$

and calculate $\varkappa_{10}$ as:

$$\varkappa_{10} = m^3c\pi(-31493793086337499488 00cm^2 + 3862312585101098558208 0c^2 - 99061808267402709 95m^5 + 7887319577691417817 2cm^3)/69374357377788174925824000 \tag{23}$$

From equation (23), we cannot proceed with further calculations. Hence $\mu_{\max}(C_{20,10}) \geq 10$, is concluded.

*3.2.2. Homogeneous Polynomials for the Class $C_{8,8}$*

*Theorem 3.7:* Let $C_{8,8}$ be the class with:

$$A(u) = (a)sin^8(u) + (b)cos(u)sin^7(u) + (d)cos^7(u)sin(u) + (e)cos^8(u),$$

$$B(u) = (g)cos^7(u)sin(u) + (h)cos(u)sin^7(u).$$

Then $\mu_{\max}(C_{8,8}) \geq 10$ is given below.

*Proof:* By using theorem 2.1, it is calculated that $\varkappa_2 = 0$, and $\varkappa_3 = \frac{35\pi(e+a)}{8}$.

For $\varkappa_3 = 0$, we substitute $e = -a$, and calculate $\varkappa_4$ as:

$$\varkappa_4 = \frac{25a\pi(h+g)}{512}.$$

If $\varkappa_4 = 0$, for $a = 0$, corollary 2.1 gives origin is centre. So, consider that $a \neq 0$. As $\pi$ is non-zero irrational, then we substitute:

$$h = -g \tag{24}$$

Furthermore, calculate $\varkappa_5 = 0$ and $\varkappa_6$ as:

$$\varkappa_6 = -\frac{315ga\pi(d+b)}{131072}.$$

If $\varkappa_6 = 0$, either $g = 0$, because $a \neq 0$, and for $g = 0$ equation (24) gives $h = 0$, which results $B(u) = 0$. According to corollary 2.2, origin is the center with mean value of $A(u)$ as zero. Therefore, by substituting

$d=-b$, we calculate $\varkappa_7=0, \varkappa_8=0$, and by proceeding further $\varkappa_9$ is as follows:

$$\varkappa_9 = \frac{g\pi(-4436293861bg^3+2121454618624a^2-766253417120b^2)}{1266637395197952}.$$

If $\varkappa_9=0$, as $g\pi \neq 0$, then we substitute an explicit function $a^2$ as follows:

$$a^2 = \frac{44362983861}{2121456118624}bg^3 + \frac{7662534117120}{21214561618624}b^2 \qquad (25)$$

Withholding equation (25), $\varkappa_{10}$ comes out as:

$$\varkappa_{10} = -g^3b\pi(26515208880883096273027072 0bg^2 + 6467219691310828765728210944b^2 + 141533397404966731868361g^5 + 32498909568340360548372 48bg^3)/ 3821675643778879581147958870016.$$

Hence we conclude that $\mu_{max}(C_{8,8}) \geq 10$.

To show the applicability of the formulas presented and clarify ideas generated in the manuscript, we have presented some examples in the next section.

## 4. Examples

In this section, some examples using series functions are given to demonstrate that the methodology used here is practical and can be used to calculate periodic solutions for various functions. We have restricted the power series to some suitable extent here.

*Example 4.1:* Consider the differential equation:

$$\frac{dz}{du} = \left(\frac{1}{1+u}\right)z^3 + (\cos u)z^2 \qquad (26)$$

For $A(u), B(u)$ we are neglecting the terms '$u^n$' for $n>4$. Like $A(u) = \frac{1}{1+u} = a - bu + cu^2 - du^3 + eu^4, B(u) = \cos u = hu - i\frac{u^2}{2!} + j_1u^4$. Here $a,b,c,d,e,h,i$ are coefficients and $j_1 = \frac{j}{4!}$. Then we calculate the periodic solutions.

*Solution:* We put $i=0$, and from theorem 2.1, we calculate:

$$\varkappa_2 = h + \frac{1}{5}j_1,$$

$$\varkappa_3 = a - \frac{1}{2}b + \frac{1}{3}c - \frac{1}{4}d + \frac{1}{5}e.$$

If $\varkappa_2=\varkappa_3=0$, we take

$$h = -\frac{1}{5}j, and\ a = \frac{1}{2}b - \frac{1}{3}c + \frac{1}{4}d - \frac{1}{5}e \qquad (27)$$

By using (27), $A(u)$ and $B(u)$ takes form as follows:

$$A(u) = b\left(u-\frac{1}{2}\right) + c\left(u^2-\frac{1}{3}\right) + d\left(u^3-\frac{1}{4}\right) + e\left(u^4-\frac{1}{5}\right) \qquad (28)$$

$$B(u) = j\left(u^4-\frac{1}{5}\right) \qquad (29)$$

In addition, we compute $\varkappa_4$ as given below:

$$\varkappa_4 = \frac{j(14d-35c+60b)}{302400}.$$

If $\varkappa_4=0$ then either $j=0$ or

$$d = \frac{35}{14}c - \frac{60}{14}b \qquad (30)$$

If $j=0$, $B(u)$ is also 0. As a result, according to corollary 2.2, the origin is a center. Assume that $j \neq 0$. If (30) holds, we calculate $\varkappa_5$ as:

$$\varkappa_5 = \frac{j^2(-56c+325b)}{6054048000}.$$

If $\varkappa_5=0$, as $j \neq 0$, we put $c = \frac{325}{56}b$, and calculate $\varkappa_6$ as:

$$\varkappa_6 = \frac{bj(21078407j^2+10315069600b)}{1261504744980480000}.$$

If $\varkappa_6=0$ then either $b=0$ or

$$b = -\frac{21078407}{10315069600}j^2 \qquad (31)$$

because $j \neq 0$. If $b=0$ then $d,c=0$; and equations (28 & 29) takes the following form:

$$A(u) = e\left(u^4-\frac{1}{5}\right),$$

$$B(u) = j\left(u^4-\frac{1}{5}\right).$$

Let $\sigma(u) = u^5 - u$ then $\dot{\sigma}(u) = 5u^4 - 1$ —also $\sigma(0)=\sigma(1)$. So, we can write $A(u) = \frac{1}{5}f\dot{\sigma}$ and $B(u) = \frac{1}{5}j\dot{\sigma}$. From theorem 2.2, origin is a center having $f(\sigma) = \frac{1}{5}f$ and $g(\sigma) = \frac{1}{5}j$. So, suppose that $b \neq 0$. By using (31), we have $\varkappa_7$ as follows:

$$\varkappa_7 = -\frac{3011201j^4(4904530106070545j^2+433150303271076864e)}{1193751333049572276388321296384000000}.$$

Recalling that $j \neq 0$ (considered above). If $\varkappa_7=0$ then we put $e = -\frac{4904530106070545}{433150303271076864}j^2$, and calculate $\varkappa_8$ as:

$$\varkappa_8 = \frac{37395284143096731929725996189267}{8601449820771049528606440089993216000000 0}j^7.$$

This $\varkappa_8$ is equal to a non-zero constant multiple of $j^7$, and that is the maximum multiplicity. Thus, the outcome of maximum multiplicity for equation (26) is even, and the sign assigned is positive. Therefore, by using remark 2.1, it can be concluded that the origin is unstable, shown in Figure 2.

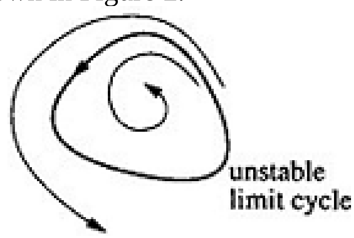


Fig. 2 Unstable origin

*Example 4.2:* Consider the differential equation:

$$\frac{dz}{du} = (\sinh u)z^3 + (\sin u)z^2 \qquad (32)$$

For $A(u), B(u)$ we are neglecting the terms '$u^n$' for $n>9$. Like $A(u) = \sinh u = cu + d_1u^3 + e_1u^5 + f_1u^7 + g_1u^9, B(u) = \sin u = iu - ju^3 + ku^5 - lu^7 + mu^9$. Here $d_1 = j_1 = \frac{1}{3!}, e_1 = k_1 = \frac{1}{5!}, l_1 = f_1 = \frac{1}{7!}, and\ g = m = \frac{1}{9!}$.

*Solution:* By following the suitable restriction of the coefficients, we put $j = k = l = 0$ . Now, using theorem 2.1 we calculate:

$$\varkappa_2 = \frac{1}{2}i + \frac{1}{3628800}m,$$

$$\varkappa_3 = \frac{1}{2}c + \frac{1}{24}d + \frac{1}{720}e + \frac{1}{40320}f + \frac{1}{3628800}g.$$

Thus multiplicity of $z = 0$ is $\mu = 2$ if $\varkappa_2 \neq 0$. In addition, $\mu = 3$ if $\varkappa_2 = 0$ but $\varkappa_3 \neq 0$. If $\varkappa_2 = \varkappa_3 = 0$, then we take $i = -\frac{2}{3628800}m$, and $c = -\frac{2}{24}d - \frac{2}{720}e - \frac{2}{40320}f - \frac{2}{3628800}g$.

By using these values, $A(u)$ and $B(u)$ takes form as follows:

$$A(u) = d\left(u^3 - \frac{2}{24}\right) + e\left(u^5 - \frac{2}{720}\right) + f\left(u^7 - \frac{2}{40320}\right) + g\left(u^9 - \frac{2}{3628800}\right) \quad (33)$$

$$B(u) = m\left(u^9 - \frac{2}{3628800}\right) \quad (34)$$

In addition, we compute $\varkappa_4$ as given below:

$$\varkappa_4 = -\frac{m(f + 105e + 3600d)}{6584094720000}$$

If $\varkappa_4 = 0$ then either $m = 0$ or

$$f = -105e - 3600d \quad (35)$$

If $m = 0$ then $B(u) = 0$ and $\varkappa_3 = 0$, shows mean value of $A(u)$ is zero. Therefore, the origin is a center. If (35) holds we calculate $\varkappa_5$ as:

$$\varkappa_5 = -\frac{m^2(14e + 1625d)}{332170212261888000000}.$$

If $\varkappa_5 = 0$, as $m \neq 0$ (taken above), we substitute $e = -\frac{1625}{14}d$, and calculate $\varkappa_6$ as:

$$\varkappa_6 = \frac{dm(-3011201m^2 + 112293973693440000d)}{5980224046090265907167232000000}.$$

If $\varkappa_6 = 0$, either $d = 0$ or

$$d = \frac{3011201}{112293973693440000}m^2 \quad (36)$$

because $m \neq 0$. If $d = 0$ equations (33) and (34) takes the form:

$$A(u) = g\left(u^9 - \frac{2}{3628800}\right), \quad B(u) = m\left(u^9 - \frac{2}{3628800}\right).$$

Origin is the center according to theorem 2.2. So, $d \neq 0$. Now, using (36), we have $\varkappa_7$ as given below:

$$\varkappa_7 = \frac{-3011201m^4\begin{pmatrix}980906021214109m^2 + \\ 1091538764243113697282g\end{pmatrix}}{1825721736634484615288993290535680000}.$$

Recalling that $m \neq 0$ (considered above). If $\varkappa_7 = 0$, we put $g = -\frac{980906021214109}{1091538764243113697282}m^2$, and calculate $\varkappa_8$ as:

$$\varkappa_8 = \frac{37395284143096719297259961189267}{19890429812315829712063531084824122478416649779624949340900000}m^7.$$

This $\varkappa_8$ is equal to a non-zero constant multiple of $m^7$ and is the maximum multiplicity.

The outcome of maximum multiplicity for equation (32) is even, and the sign assigned is positive. So, by using remark 2.1, it can be concluded that the origin is unstable, shown in Figure 2.

## 5. Conclusion

Periodic solutions by using the bifurcation method are found for the first-order cubic non-autonomous differential equation here. We discussed two types of coefficients; algebraic and trigonometric (non-homogeneous and homogeneous) coefficients for various classes. The scientific novelty lies under the fact that in literature, we do not have maximum periodic solutions as ten, so we used our newly developed formula [3] to find multiplicity ten for classes C18,9, C20,10 with non-homogeneous and C8,8, with homogeneous trigonometric coefficients, while C10,1, and C10,2 with polynomial coefficients. We got maximum multiplicity for classes C10,1 and C10,2 with polynomial coefficient and C8,8, C20,10 with trigonometric coefficient as 10, which is a challenge and concept prescribed in the second part of the Hilbert sixteenth problem. The stability of limit cycles of the classes mentioned above is also discussed briefly. We see that [2,13] have studied the periodic solutions, and they found the maximum periodic solutions 8, but we have calculated 10. We used computer algebra Maple 18 to handle the complicated and lengthy computation of integrals while calculating the limit cycles. We used the systematic bifurcation method that occurs under the perturbation of the coefficients. In addition, we have added some examples in section 5, which shows that the techniques used in this article are correct and supportive. Future work can generalize the same periodic solution concept by generalizing theorem 2.1 to calculate maximum multiplicity greater than ten.

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