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# Exploring Trigonometric Treasures: A Dynamic Approach to Teaching the Sine and Cosine Functions Using Realistic Mathematics Education and GeoGebra - A Case Study

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**Abstract:** This study is situated within the framework of realistic mathematics education (RME) and presents a classroom proposal aimed at illuminating and modeling everyday periodic phenomena. Emphasis is placed on recognizing periodic functions as fundamental tools for characterizing the sine and cosine trigonometric functions. The investigation uses the concept of mathematization levels to delineate the transition from informal to formal knowledge through real-world problem-solving using the GeoGebra platform. This research was conducted with tenth-grade students at the San Vicente Educational Institution in the Buenaventura District. The outcomes reveal that students engaged in activities, and subsequent analyses demonstrate a nuanced understanding of periodic phenomena within realistic contexts. This comprehension encompasses proficiency in various representation systems, including tabular, graphical, and algebraic methods. Moreover, the process of progressing from informal to formal knowledge is marked by incremental levels of mathematization, encompassing both horizontal and vertical processes. The main focus of this research is to characterize the process of mathematization within a realistic context, particularly concerning the sine and cosine trigonometric functions, and evaluate the strengths and limitations of this approach. This study underscores the potential of RME for elucidating intricate mathematical modeling processes within real-world contexts. These insights underscore the value of integrating authentic scenarios to foster a comprehensive understanding of mathematical concepts. Challenges are also acknowledged, particularly in achieving the formalization of generalized models. Thus, this research contributes to discussions on effective mathematics education by showcasing the transformative potential of contextualized pedagogical strategies.

**Keywords:** realistic mathematics education, trigonometric functions, levels of mathematization, context, levels of understanding.

# 探索三角宝藏：使用现实数学教育和地理几何教授正弦和余弦函数的动态方法-案例研究

**摘要**：这项研究位于现实数学教育（RME）的框架内，提出了旨在阐明和模拟日常周期性现象的课堂建议。重点放在认识周期函数作为表征正弦和余弦三角函数的基本工具。该调查使用数学化水平的概念来描述通过使用地理几何平台解决现实世界问题从非正式知识到正式知识的转变。这项研究是在布埃纳文图拉区圣维森特教育机构的十年级学生中进行的。结

果表明，学生参与了活动，随后的分析表明，他们对现实背景下的周期性现象有细致入微的理解。这种理解涵盖了对各种表示系统的熟练程度，包括表格、图形和代数方法。此外，从非正式知识发展到正式知识的过程以数学化水平的递增为标志，包括横向和纵向过程。这项研究的主要重点是在现实背景下描述数学化过程，特别是关于正弦和余弦三角函数，并评估这种方法的优点和局限性。这项研究强调了RME在阐明现实世界中复杂的数学建模过程方面的潜力。这些见解强调了整合真实场景以促进对数学概念的全面理解的价值。我们也承认存在挑战，特别是在实现广义模型的形式化方面。因此，这项研究通过展示情境化教学策略的变革潜力，有助于有效数学教育的讨论。

**关键词**：现实数学教育、三角函数、数学化水平、背景、理解水平。

## Introduction

This research aligns with the didactics of mathematics in basic education. It operates within the framework of the realistic mathematics education (RME) approach and characterizes the process of mathematization through problem-solving situations in a realistic context. This approach facilitates the transition from informal to formal knowledge when working with the sine and cosine trigonometric functions using the GeoGebra program.

This investigation studies periodic and variation phenomena mathematically represented through trigonometric functions such as sine and cosine. Given their periodic nature, these functions are conducive to the mathematization of such phenomena. This perspective is supported by research such as that by Buendía [2] and Montiel [2], which highlights periodicity as a fundamental analytical property of trigonometric functions, making this property meaningful in educational contexts.

This study aims to transform the teaching of trigonometric functions in the classroom, which, according to [3], has often been limited to a static model, primarily focusing on trigonometric ratios and neglecting the study of variations and covariations of trigonometric functions, including the periodicity property.

In this research, we first introduce the general aspects, outlining the problem, emphasizing student difficulties in learning mathematics, and focusing on the research object, the sine and cosine trigonometric functions.

Second, we delve into the theoretical references that guide the fundamental criteria for the proposal's design, encompassing didactic, curricular, mathematical, and technological aspects.

Next, we introduce the research's methodological framework, detailing the research method, design, data collection instruments, participants, and the development of interventions and interactions with students in the classroom.

Finally, we present the results of the tasks performed by the participating students, along with the conclusions and reflections about the specific objectives, theoretical framework, and analysis of each student's production.

## 1. Description and Problem Formulation

Learning mathematics presents numerous challenges for students, which can stem from the nature of the discipline itself (mathematical concepts and thought processes), the teaching process, students' cognitive processes, or a lack of a rational attitude toward mathematics. These difficulties can be addressed from various perspectives, focusing on student cognitive development, mathematics curriculum, and teaching methods [4].

To overcome these challenges and enhance mathematics understanding, various methodological strategies have been developed. Some researchers propose that mathematics education should focus on mathematical processes such as problem-solving and mathematical modeling. This approach emphasizes more humanized mathematics, making it less formal and more meaningful for students [5].

As a result, the teaching of mathematics must transform from a teacher-centered approach to a student-centered one. This shift entails changes in the roles of both students and teachers. Teachers become facilitators and guides, and students become active agents in constructing mathematical knowledge [6].

In this context, the Ministry of Education of Colombia (MEN) [7] emphasizes relating learning content to students' everyday experiences and teaching them in the context of problem situations and viewpoint exchange. Mathematical modeling, an essential aspect of problem-solving, allows students to construct their models at different levels of complexity, connecting mathematical objects with real-world contexts [8-10].

However, introducing mathematical modeling in the classroom can be challenging. Constructing a model is not automatic and requires time for students to apply mathematical knowledge, context understanding, and skills to represent relationships between quantities [11].

This research proposes a classroom approach using the sine and cosine trigonometric functions, which, due to their periodic nature, can model various repetitive phenomena such as tides, vibrations, sound waves, and bicycle wheel motion. These situations facilitate students' mathematical model building, connecting the real world with abstract mathematical concepts.

Despite the advantages of trigonometric functions, especially Sine and Cosine, in modeling real-world phenomena, their use in education often falls short. These functions are typically taught as extensions of trigonometric ratios, neglecting their origins and practical applications [2].

Therefore, this study proposes an RME-based classroom approach for teaching the sine and cosine trigonometric functions. It seeks to characterize how students transition from informal to formal knowledge through problem-solving situations, bridging the gap between the real world and abstract mathematical concepts, with the aid of the GeoGebra software.

Addressing these elements, this study addresses the following question: what are the characteristics of the mathematization process in tenth-grade students when problem-solving situations in a realistic context are introduced to facilitate the transition from informal to formal knowledge with the sine and cosine trigonometric functions using Geogebra?

## 2. Justification

The National Ministry of Education outlines in its standards the importance of describing and modeling real-world periodic phenomena using trigonometric functions [8-10]. Hence, incorporating mathematical modeling into teaching trigonometric functions is relevant because it connects mathematical objects with real-world contexts. The MEN [7] emphasizes relating learning to students' daily experiences and presenting it within problem-solving contexts and viewpoint exchange.

Despite these guidelines and the efforts of researchers such as [2] and [3], trigonometric functions are still often taught as extensions of trigonometric ratios in schools. Their use in modeling periodic phenomena for mathematical construction or problem-solving purposes is often underemphasized.

This study develops a classroom proposal for teaching the sine and cosine trigonometric functions from a realistic perspective. The approach intertwines context and mathematical activity, highlighting essential aspects of these functions, such as periodicity, variation, and amplitude. The goal of this study is to provide insight into the role of realistic contexts in mathematical modeling and expand the theoretical and methodological horizons of curriculum proposals for mathematics.

Regarding the use of ICT in designing and implementing the proposed activities, the mathematics curriculum guidelines recognize the impact of new technologies on curricular emphases and applications [7]. Technology transforms the way mathematics is perceived in classrooms [12].

In the design and execution of activities, the dynamic geometry software Geogebra is used as a pedagogical resource for classroom mediation and mathematical knowledge construction. Geogebra enables multiple representational shifts, allowing students to work with graphical, algebraic, and tabular representations concurrently. This aligns with the MEN's emphasis on using different representation modes to express mathematical ideas [8-10].

By incorporating Geogebra, this proposal contributes to mathematics education, specifically the teaching and learning of the sine and cosine trigonometric functions from a realistic perspective. This approach bridges the gap between the real and abstract, characterizing phenomena relevant to Buenaventura, such as tidal behavior.

## 3. Background

The related studies cited provide valuable insights into enhancing the teaching and learning of trigonometric functions through mathematical modeling and technology integration. [13] highlighted the significance of visualization and Geogebra in aiding students' understanding of the properties of these functions. [14] underscored the role of mathematical modeling, particularly within the RME framework, in connecting mathematics with real-world contexts. [15] emphasized the value of technology in creating simulations and graphical representations to enhance the comprehension. [16] integrated mathematical modeling and ICT to effectively teach these functions. In summary, these studies collectively advocate the synergistic use of modeling and technology in trigonometry education, aligning with the goals of this proposal.

## 4. Theoretical Framework

This work presents a classroom proposal for teaching the sine and cosine trigonometric functions based on the levels principle of RME, integrating Geogebra. Therefore, this chapter introduces some theoretical references that are deemed relevant for the selection of fundamental criteria in the design of this proposal. These are developed in four phases.

First, a general overview is provided regarding the main theoretical contributions of RME, grounded in the theories of Hans Freudenthal [17, 18], in terms of its origins, concepts of teaching and learning mathematics, and principles. Second, a description of mathematical modeling is presented from the curricular guidelines and basic mathematics standards. In the third phase, historical contributions are explored based on [2], which sheds light on the evolution of conceptions around the sine and cosine trigonometric functions as they developed into mathematical objects. Concurrently, a mathematical description of these functions is supported by [19]. Finally, a panorama of the use of technology in mathematics education is presented as a pedagogical resource [20].

## 4.1. Didactic Reference

### *4.1.1. RME: Its Beginnings*

This section on the origins of RME and its incorporation into different mathematics curricula as a new approach to teaching and learning has been developed considering [17]. In this study, a detailed description of the historical processes regarding the teaching and learning of mathematics in the 1960s is provided. This period marked a reform or change in the mathematics curricula of schools of that era based on the proposals of a group of mathematics education researchers in the Netherlands, led by Hans Freudenthal.

In the years leading up to the 1960s, after the end of World War II, several countries established compulsory and equal education up to a certain age determined by the government. Educators, mathematicians, and psychologists recognized the difficulties students faced in learning mathematics. These difficulties were exacerbated if students lacked economic support from their families for private tutoring. Despite the inadequacy of teaching methods for students' intellectual and psychological abilities, there was no change in the conventional methods of teaching mathematics. Moreover, the way mathematical content was structured in curricula was overly abstract.

The launch of the first Soviet artificial satellite in 1957 sparked concerns in Western countries, particularly the United States, about their students' mathematical education. They realized that to achieve high levels of technology, they needed to improve mathematics education in secondary schools. For them, a country capable of achieving such advancements required scientists with a high level of mathematical education [17].

This concern led them to convene mathematics specialists and secondary education teachers. The premise was to determine the most appropriate mathematics to produce scientists capable of advancing science and technology. Thus, the United States embarked on a search for tools or strategies that could guide changes in their education system, as it was clear that the way teaching processes were conducted was inadequate to achieve their goals.

According to mathematicians Choquet, Stone, and Dieudonné, the problem lay in the gap between secondary school and university. Secondary school programs in various countries were disconnected from modern mathematical concepts developed within universities. This disconnect created significant difficulties in students' understanding, leading to disinterest. The solution proposed by several countries was to shift from teaching Euclidean geometry to introducing set theory and algebraic structures (modern mathematics) in school classes. However, the rigor of the mathematical content remained unchanged and was incomprehensible to many students.

This transition to modern mathematics, according to [20], did not solve the problem of overcoming students' difficulties in learning mathematics. On the contrary, it continued to be regarded as mathematics not for everyone but for a specific group (selective), precisely because its rigor remained unchanged. Students who lacked access to private tutoring found it impossible to continue their studies. This educational approach tended to serve an elite group.

In response to this situation, the Wiskobas project was initiated in the Netherlands in 1968. Led by a group of mathematics educators from primary and secondary levels under the direction of Hans Freudenthal, this project was conducted at the Institute for the Development of Mathematics Education (IOWO) at Utrecht State University. The project was conducted between 1970 and 1977, replacing textbooks based on modern mathematics in schools.

Freudenthal developed a curriculum project for teaching elementary mathematics to innovate mathematics education at the national level. This involved training in-service teachers to act as agents of change.

For Freudenthal [21], a central concern regarding the educational and academic reality of his time was as follows: "We need to decide urgently whether the image of mathematics is for an elite or everyone, a picture of mathematics for the whole education". He believed that all students needed to have some form of contact with mathematical activity as a structuring or organizing activity of mathematization accessible to all human beings [21, 22], in contrast to the approach of modern mathematics.

As a result, the theory of RME gained traction in the Netherlands. It was primarily developed on the basis of Freudenthal's ideas [21, 22] and emerged as a response to the modern mathematics movement and the mechanistic approach to teaching mathematics prevalent in Dutch schools at that time.

### *4.1.2. Key Principles of RME*

RME is rooted in the belief that mathematics should be taught in a way that reflects its actual use and significance in solving real-world problems.

One of the key principles of RME is the idea of "mathematizing reality." This means that mathematical concepts should be developed from students' experiences and everyday situations. Instead of presenting abstract mathematical ideas in isolation, RME encourages teachers to engage students in exploring and solving contextualized problems. By anchoring mathematics in real-life contexts, students can better understand the relevance and utility of mathematical concepts.

Another central principle of RME is the notion of "guided reinvention." Freudenthal believed that students should be guided to rediscover mathematical ideas and relationships on their own. Teachers act as facilitators, providing carefully designed tasks and questions that lead students to construct mathematical knowledge through exploration and problem-solving. This approach fosters a deeper understanding of mathematical concepts as students actively participate in the process of discovery.

Furthermore, RME emphasizes the importance of variation in problem-solving. Instead of presenting a single algorithm or method for solving a particular type of problem, RME encourages teachers to offer diverse problem-solving strategies and scenarios. This helps students develop a flexible and adaptable mathematical mindset, allowing them to tackle problems creatively.

Collaboration and communication are also central to RME. Students are encouraged to discuss and share their mathematical ideas and approaches with their peers. Through collaborative activities, students learn to articulate their reasoning, critique others' ideas, and refine their understanding.

Incorporating technology and real-world tools is another aspect of RME. The use of tools such as Geogebra, which allows dynamic visualization and exploration of mathematical concepts, aligns with RME's emphasis on engaging students in meaningful mathematical experiences.

### *4.1.3. Theoretical Foundations of RME*

RME does not aim to be a comprehensive theory of learning; rather, it serves as a comprehensive theory focused on the development of mathematical tools and conceptual understanding of problem-solving. RME is based on the following key ideas:

*Starting from context and realistic problem situations:* As stated by Freudenthal [22], "A context is that domain of reality which, in a particular learning process, is revealed to the student to be mathematized". According to this perspective, mathematics has historically emerged as a tool for mathematizing situations from the natural and social environment. Consequently, mathematics education should be rooted in organizing such situations. The goal is for students to construct or reinvent mathematical tools based on real-context problems [21, 22].

*Use of student-generated models:* RME encourages the use of models—such as materials, diagrams, and symbols—that emerge from students' mathematical activities. These models serve as tools for representing and organizing contextual situations.

The model serves as an intermediary, often indispensable, through which a complex reality or theory is idealized or simplified to become amenable to formal mathematical treatment [22]. In RME, the term "model" does not refer to pre-established models from formal mathematics (models for situations), but rather to models that emerge from contextual situations (situation for the model). In the teaching and learning process, problem situations are presented, leading students to engage in organizing and reorganizing activities that produce models. Students transition from specific situations to formal, generalizable models that can be applied to various contexts and situations. This shift moves from a "model of" a specific situation to a "model for" mathematical reasoning in diverse situations, both within and outside mathematics [18].

In this theory, models are considered objects of work and reflection. Actions and operations are performed on these models, and they are used to visualize, explain, compare, contrast, and verify relationships. To fulfill these purposes, these models must satisfy the following conditions:

1. Be rooted in realistic and imaginable contexts;
2. Be sufficiently flexible to be applied at a more advanced or broader level. This implies that the model should support vertical mathematization progression without inhibiting the possibility of returning to the situations from which the strategy originated. In other words, students should always be able to revert to lower levels, which makes the models highly powerful;
3. Be viable. The models should behave in a natural, self-evident manner. They should align with students' informal strategies as if the students could have reinvented the models themselves. In addition, the models should be easily adaptable to other situations.

The search for contexts and models that naturally lend themselves to mathematization aligns with what Freudenthal [21] refers to as didactic phenomenology. This approach draws upon the history of mathematics and students' independent productions and constructions that emerge during the teaching process [23].

*Recognizing the key role of the teacher as a guide and organizer in the classroom:* According to Freudenthal [22], mathematics teaching should take the form of guided reinvention. This involves students re-inventing mathematical ideas and tools by organizing or structuring problem situations in interaction with their peers and under the guidance of the teacher. In this interactive teaching, students are asked to explain, justify, agree or disagree, question alternatives, and reflect on them.

The teacher holds a well-defined role as a mediator between students and the problem situations at hand, between students themselves, and between the informal productions of students and the formal tools of mathematics that have already been institutionalized as disciplinary norms.

*Learning mathematics as a social activity leading to higher levels of understanding:* Learning mathematics is considered a social activity in which collective reflection leads to higher levels of comprehension. Both vertical (teacher-student) and horizontal (student-student) social interactions play a central role. The teacher's management of these events maximizes opportunities for students to produce, exchange, and assimilate ideas. This concept counters a uniform pedagogical process, instead focusing on individual student paths. It supports maintaining the overall class structure as a unit of organization and promotes cooperative work in heterogeneous groups. This approach, advocated by Freudenthal since the 1940s, ensures that problems are selected to yield solutions that appeal to different levels of understanding, allowing all students to engage with the material.

*Strong interrelation and integration of mathematical curriculum components:* Resolving realistic problem situations often necessitates establishing connections and applying a broad range of mathematical understandings and tools. RME avoids deep distinctions between curriculum components, which provides greater coherence to teaching and enables different ways of mathematizing situations through various models and languages. This high level of coherence is achieved throughout the curriculum [24, 25].

One reason mathematics can become challenging is when topics are taught in isolation, neglecting the connections and interrelationships between them. In applications, the use of different mathematical components is necessary. Merely employing arithmetic or algebra or relying solely on geometry is insufficient for solving problems.

Furthermore, RME introduces the following principles concerning mathematics teaching and learning, as outlined by [1].

Table 1 Theoretical design of the RME approach [1]

| **General Theory of RME** | |
|---|---|
| **What** | **How** |
| • Significant human activity<br>• Horizontal and vertical mathematization<br>• Low and high levels of skills | *Teaching principles:*<br>• Reality<br>• Interconnection<br>• Reinvention<br>*Learning principles:*<br>• Activity<br>• Levels<br>• Interaction |

*Activity:* This principle is rooted in Freudenthal's notion of mathematics as a human activity aimed at organizing (mathematizing) the surrounding world, including mathematics itself [21, 22]. Freudenthal's primary goal is to provide an explanation or approach to everyday reality through mathematical models, where the informal knowledge possessed by students serves as the starting point for mathematization or the creation of a mathematical model.

*Reality:* From the perspective of RME, learning mathematics means engaging in mathematical activity, a "reflective mental activity" [22], in which solving problems situated in realistic contexts, in the sense of achievability or imaginability, is central to the task of mathematization. However, the term "realistic" doesn't solely refer to a connection with the real world; it also pertains to problem situations that are real in students' minds [1]. The context of problems presented to students can be from the real world, but this is not always the case [26]. This implies that although initially, problems are selected from the same context, students must later detach from it to acquire a more general nature (progressive mathematization) transforming them into mathematical models.

*Level:* transition from informal to formal knowledge.

In RME, the learning of mathematics occurs at some point between informal mathematics (related to context) and formal mathematics. Freudenthal and his collaborators studied how students transition from informal to pre-formal and from there to formal knowledge and how to assist them in this process.

Freudenthal's main contributions revolved around transforming the paradigm of challenging and hard-to-grasp mathematics, which was accessible only to a specific group, into a more humanistic form of mathematics. He emphasized the close relationship between everyday situations and formal mathematics (concrete-abstract), focusing on the process of mathematization and the formulation of classroom activities and interactions between students and between the teacher and students. This considered various uses of the knowledge to be taught and different ways students appropriate this knowledge.

Within this process of progressive mathematization, RME considers that students go through different levels of comprehension characterized by various types of mental and linguistic activities. These levels are situational, referential, general, formal, and linked to the use of strategies, models, and languages of different cognitive categories, without constituting a strictly ordered hierarchy [21, 22, 25].

When a student progresses from one level to another, their development is reflected in the subsequent level, as they are tested in that new level. Furthermore, all these levels are closely related, as affirmed by Freudenthal [21]: "The evolution between levels occurs when the activity at one level is subjected to analysis in the next, and the operative theme in one level becomes the object of the next level".

These levels materialize into two processes of mathematization: horizontal and vertical. The former is characterized by the utilization of informal or pre-formal knowledge, through which students (assisted by the teacher) create a model of a specific situation using some form of mathematics, thus progressing to different levels of abstraction [27]. The latter is marked by model refinement, conceptual schematization, and gradual formalization, resulting in higher levels of mathematical formalization [18]. According to [28], this involves transforming a contextual problem into a mathematical one.

*Situational level:* Situated within horizontal mathematization, this level focuses on interpreting the problem situation and using strategies entirely linked to the context. The student's informal or pre-formal knowledge, experience, and strategy for mathematically describing and interpreting a problem are considered fundamental. Emphasis is not placed on the use of formal mathematical models.

The subsequent levels pertain to vertical mathematization as they emphasize formula derivation, proof application, generalization, and so on.

*Referential level:* Here, graphic, material, or notational representations emerge, along with personal descriptions, concepts, and procedures that schematize the problem. Models are considered "models of" as they refer to specific situations from which they originated.

*General level:* Developed through exploration, reflection, and generalization of what appeared in the previous level, this stage provides a mathematical focus on strategies that surpass contextual reference. Reflection on concepts, procedures, strategies, and models used at the prior level leads to generalizable aspects. Students can conclude that these aspects apply to sets of analogous problems, leading to models for their resolution.

*Formal level:* At this stage, students comprehend and work with conventional concepts, procedures, and notations specific to the branch of mathematics they are working with.

These levels are dynamic, and students can function at different comprehension levels for various content or aspects of the same content. However, for these levels to function effectively, they must be rooted in concrete and flexible situations to be highly usable at higher levels during the mathematization process.

*Guided reinvention:* This principle centers on the active involvement of the teacher in guiding the student's construction or reinvention of mathematical knowledge through discussions, allowing students to compare, explain, and present different strategies used in solving similar problems. This process guides a potential learning path toward progressive mathematization [29].

*Interaction:* Interaction between students (peer-to-peer) and between students and teachers is essential in realistic didactics. Explicit negotiation, intervention, discussion, cooperation, and evaluation are essential elements in a constructive learning process where students' informal methods serve as a starting point for formal methods.

*Interrelation:* This principle underscores the centrality of interconnectedness among mathematical content units or themes. The integration of content from various learning axes should be woven into problem situations posed in the classroom. This fosters the combined use of arithmetic, algebra, and geometry to solve problems, rather than teaching them as isolated entities. As per Freudenthal [22], the interrelation between axes should occur as early, frequently, and for as long as possible.

## 5. Curricular Reference

Within the Curriculum Guidelines (1998) and Basic Competency Standards (2006), mathematical modeling or mathematization is one of the five processes of mathematical activity, defined as "the detection of patterns that repeat in everyday, scientific, and mathematical situations to mentally reconstruct them" [8-10].

Mathematization aims to imbue school mathematics with a broader sense that enables students to apply their knowledge within and beyond the school setting, in contexts where they can formulate hypotheses and make decisions to address and adapt to new situations [30].

The MEN [7] affirms that "it is necessary to relate learning contents to students' everyday experience and present and teach them in the context of problematic situations and exchange of viewpoints". Thus, mathematization is considered to be a crucial element in mathematics classes.

Modeling allows students to observe, reflect, discuss, explain, predict, review, and thereby construct mathematical concepts meaningfully. Consequently, it is considered that all students need to experience mathematization processes that lead to the discovery, creation, and use of models at all levels [7].

Hence, in the teaching and learning process related to trigonometry and modeling, the Basic Competency Standards in Mathematics declare as a purpose at the end of the secondary education cycle that students should be able to "describe and model periodic phenomena of the real world using trigonometric relationships and functions". This indicates that teaching trigonometry can also relate mathematics to students' contexts. Furthermore, trigonometry is embedded in the development of spatial thinking, although it is not exclusive to this type of thinking and contains elements of numerical, variational, and metric aspects.

Finally, the Basic Competency established a reference point for changing the way trigonometry can be taught, as learning trigonometric functions enables the modeling of "real-life" situations within the context of mathematics. This allows the expansion of the field of study to the analysis of lengths, the study of periodic and cyclical motions (e.g., spring motion, tide behavior, wheel rotation), and addresses content present in higher education with real problems encountered there.

## 6. Research Methodology

The aim of this study is to characterize the process of mathematization through problem situations in a realistic context, facilitating the transition from informal to formal knowledge, specifically focusing on working with trigonometric functions. The research method consists of design, data collection instruments, participants, and execution of each situation, including intervention and interaction with students in the classroom. This approach supports the achievement of the stated objective.

The chosen methodology is qualitative as it enables a deep exploration of phenomena, their dynamic structures, and a detailed explanation of behaviors and manifestations within these phenomena. The constant interaction between students and between students and teachers during the development and subsequent analysis of the proposed activities using the dynamic geometry software Geogebra is vital. This interaction aligns with the theories outlined in the theoretical framework.

Barrio and Córdoba [29] strongly emphasize qualitative methods for studying phenomena related to the mathematical modeling process. These methods enable the direct sourcing of data from natural environments and emphasize descriptive elements and manifestation processes rather than just the final results.

Considering the purpose of this study, a qualitative research approach was chosen. Within this approach, the case study method is employed as it aligns well with the research objectives and methodology.

### 6.1. Case Study Method

The research methodology employed is qualitative, as defined by [31], who describe it as a method of learning about a complex situation (such as a classroom in a school), achieved through a comprehensive understanding of the situation. This understanding is obtained through description and analysis within the context of the situation as a whole.

Similarly, [32] defines a case study as a qualitative methodology of group analysis that allows for concluding real or simulated phenomena in a formative experimental, research, and/or human personality development context, or any other individualized and unique reality.

Furthermore, a case study can be classified as descriptive, aiming to identify key elements or variables impacting a phenomenon; explanatory, seeking to uncover connections between variables and the phenomenon while providing sufficient theoretical rationale to observed relationships; and predictive, examining the boundary conditions of a theory [33].

Considering the above and in line with the focus of this research, descriptive and explanatory studies were chosen. In terms of the descriptive aspect, real-life situations will help identify and describe the processes students undergo when modeling situations using trigonometric functions (sine and cosine) with the aid of Geogebra. This is done on the basis of the RME principles, particularly focusing on levels of mathematization. In terms of explanatory aspects, the effectiveness of the RME theory is examined through the development of activities given to the selected students for this study.

#### *6.1.1. Case Study Design*

According to [33], it is essential to specify in any research how information related to constructs will be collected. This involves clarifying various sources from which information will be obtained and instruments used for data collection. Subsequently, a logical link between the obtained data and the propositions should be derived. Finally, presenting research results through a series of conclusions should strengthen the theories or frameworks embedded in the research’s theoretical framework. Consequently, the following will be described in detail: the research context or location, information collection techniques, the unit of analysis, the activities conducted with the students, and the predictive analysis of situations.

The study was conducted in a public institution located in the continental zone of Buenaventura District. The institution has around 2000 students. Notably, it offers education to girls only and covers preschool, elementary school, middle school, high school, and adult education in cycles.

The institution’s mission, as part of its Educational Institutional Project, is to provide comprehensive education, emphasizing analytical and critical thinking, ethical, moral, and spiritual values, and enabling students to develop knowledge and skills with a humanistic constructivist approach. This equips them to effectively address scientific and technological challenges in a multicultural and globalized world.

The institution has computer labs fully with computational equipment, which facilitated this study. Therefore, the use of technological equipment was essential for the proposed activities.

The case study did not involve selecting a representative sample from a population but rather a theoretical sample. "The purpose of a theoretical sample is to choose cases that are likely to replicate or extend the emerging theory. The number of cases should be added until theory saturation is reached" [33].

Considering the above, this study’s participant subjects were 10 tenth-grade students from the Institution, selected based on their strong academic performance, particularly in the field of mathematics. This selection potentially replicated the theoretical framework of the research.

*6.1.2. Data Collection Techniques and Instruments*

The case study follows a common ethnographic methodology for studying equally common settings (such as a classroom). This involves a direct study of individuals or groups over a certain period, using participant observation or interviews. This approach unveils the meanings underlying the actions and interactions that constitute the social reality of the studied group [31]. Accordingly, the technique used for data collection was participant observation.

*Participant Observation*

Observation is the primary tool for obtaining research results in line with the stated objectives. Specifically, it examines how and why students construct mathematical models, considering levels of mathematization, through interactions between themselves and the researchers. It also allows a detailed description of each process (conjectures, discussions, attempts) that the students undertook to mathematically model real-world situations proposed in each task.

*Students' Written Productions*

These are a fundamental part of this work as they contribute to the analysis, considering each of the described categories of analysis (vertical mathematization and horizontal mathematization). They provide insight into the effectiveness of the theory developed in the research through results and conclusions.

*Technological Resources*

As previously mentioned, this study involved the use of software (Geogebra) installed on computers in one of the institution's rooms for activity development. For data collection, a camera was used to capture evidence of the work done by the students.

## 6.2. Interaction with Students

The practice was conducted in four morning sessions lasting for 2–3 h depending on the intensity of the tasks. These sessions took place in one of the institution's computer labs because the Geogebra software was required. The exception was the first session, in which software usage was unnecessary for the tasks at hand. Notably, in the first session, within ten minutes, the students introduced themselves, sharing their names and respective tenth-grade affiliations. This was because the group consisted of students from each of the 10th grades within the institution. An overview of the project's objectives was provided, followed by the continuation of the planned activity for the session.

Table 2 Schedule of the task execution (Elaborated by the authors)

| Levels of Understanding | Session | Task | Duration(min.) |
|---|---|---|---|
| **Situational** | 1 | Recognize Periodic and Variation Phenomena | 120 |
| **Referential** | 2 | Approach the Functions Sine and Cosine | 180 |
| | 3 | Approach the Functions Sine and Cosine | 120 |
| **General and Formal** | 4 | Generalization of the Trigonometric Functions Sine and Cosine | 180 |

## 6.3. Category of Analysis

The analysis of information in this research focuses on the productions and interactions of the participating students, interpreted through the theoretical and methodological elements of RME, which account for the objectives outlined in the research. The analysis of information is not simply based on a description of what was observed or a transcription of the opinions and written productions of the students. The key is to give meaning and find relationships in what the students and teachers think, say, do, and construct when engaged in modeling processes in the classroom [34].

The analysis of this research will specifically emphasize the principle of levels or levels of understanding, where the process of mathematization through problem situations in a realistic context will be characterized. This is done in a way that facilitates the transition from informal to formal knowledge by the participants when working with the sine and cosine trigonometric functions using Geogebra. Considering that constructing a model is not automatic or immediate, but rather a complex process, the analysis is grounded in the levels of understanding as they help track the learning processes, both in terms of the work done by the students in horizontal and vertical mathematization levels.

In this regard, the methodological order for analyzing the results of the students' productions, once the information has been collected, involves a predictive analysis (initial assumptions about how students intend to solve the problem situation). These predictions serve as a basis for creating categories of analysis, which are further enriched by visible elements in the participants' productions. Finally, a prospective analysis is presented to compare what was stated in the predictive analysis with what the participants actually did and constructed interactively.

In this way, a description will be provided on how the students constructed the expected mathematical models, alongside difficulties they encountered in transitioning from informal to formal knowledge.

**6.4. Design of the Situations**

Considering the theoretical elements and methodology proposed by RME, specifically in the principle of levels, mathematical situations related to the sine and cosine trigonometric functions have been designed. The purpose of these situations is to characterize the process of progressive mathematization through a series of guiding questions and the researchers' guidance. All the situations stem from everyday life problems or the students' context and are related to different levels of horizontal mathematization (situational level) and vertical mathematization (referential level and general level).

In this regard, Situation 1 ("Recognize Periodic and Variation Phenomena") is framed within the situational level, where students introduce their informal knowledge and situational strategies to identify and discover the underlying mathematics in that scenario. Situation 2 ("Approach Trigonometric Sine and Cosine Functions") is situated at the referential level, where students use various forms of representation (graphical, tabular, and algebraic) to create a mathematical model of a particular situation. Finally, Situation 3 ("Generalization of Trigonometric Sine and Cosine Functions") is positioned at the general and formal levels. Here, students, based on their exploration at the previous level, begin making generalizations that can be applied not only to a specific situation but also to a set of problems. This leads to the creation of models for solving these problems while also understanding and using the concepts, procedures, and conventional notations inherent to the context-linked mathematics addressed.

It is crucial to clarify that the fundamental purpose of the designed situations is not solely to enhance learning perspectives of the sine and cosine trigonometric functions, although this might be an outcome. The main objective is to characterize the mathematization processes in a way that facilitates the transition from informal to formal knowledge when students work on constructing models using the sine and cosine functions. Each situation is placed within the process of horizontal (situational level) and vertical (referential, general, and formal levels) mathematization.

**6.5. Predictive Analysis of Situations**

In this section, some possible mathematical models that may arise during the tasks executed by the participating students in the research are discussed. These models are associated with each of the defined levels of horizontal and vertical mathematization in the RME framework. The goal is to identify and define certain characteristics that delimit the levels of mathematization when working with the designed tasks. This analysis provides a prospective insight into how students use their informal knowledge to produce models and how these models evolve within the realm of mathematics.

The models emerge from the students' constructive activity, meaning that there is no pre-constructed or formally imposed model. This emphasizes the value of learners' productions in promoting the emergence of specific solutions that can be schematized and have a vertical perspective. Thus, the analysis revolves around possibilities, as different models may emerge during the application and interaction with the students.

Therefore, anticipating these models is crucial for establishing contexts that allow for a diversity of models with vertical perspectives. However, it is important to clarify that the separation and categorization of models into a specific level of mathematization represents a theoretical extension that can aid in understanding the process of mathematical modeling. This is because the horizontal and vertical components are not developed independently or sequentially; rather, they continuously interweave.

*Situation 1: Recognizing Periodic and Variation Phenomena*

In this situation, students are expected to use strategies tied to the context of the situation. This primarily involves visually organizing data, schematizing, and searching for regularities and non-mathematical relationships. Two tasks are proposed for this situation: Task 1 (Understanding the Situation) and Task 2 (Approaching Variation and Periodicity Phenomena). These tasks specifically emphasize the use of verbal statements and tabular representation of the studied phenomenon (tidal behavior). This helps students understand the situation and identify regularity patterns, thus characterizing variation and periodicity phenomena within realistic contexts.

Fig. 1 Students watching a video "The Tides" (Elaborated by the authors)

*Situation 2: Approaching the Trigonometric Sine and Cosine Functions*

This situation specifically emphasizes the use of different forms of representation (tabular, graphical, and algebraic) of the trigonometric sine and cosine functions in specific terms. Three tasks were designed for this purpose: Task 1 (Understanding the Situation), which emphasizes students' verbal productions for a general comprehension of the situation (bicycle wheel), Task 2 (Tabular and Graphical Representation of a Periodic Function), and Task 3 (Approximations to Algebraic Expressions). These tasks involve interpreting and creating graphs on the Cartesian plane and establishing dependencies and independencies of the functions. This allows algebraic representation of the sine and cosine functions for a specific situation.

*Situation 3: Generalization of the Sine and Cosine Trigonometric Functions*

Given its level (general and formal), this situation considers mathematical elements such as circumference length, periodicity calculation, amplitude, phase shift, variation, and dependence between the magnitudes (height and time) of the sine and cosine trigonometric functions. This incorporates various representations (tabular, graphical, algebraic, Cartesian) in line with the developments from the previous levels. Two tasks are proposed: Task 1 (Identifying Sine and Cosine Function Patterns from Circular Motion), where students are expected to identify patterns of the sine and cosine functions (periodicity, variation, amplitude), allowing them to create a general algebraic model representing the situation (the Ferris wheel). The expected models may be $h(t) = A\sin(B\ t)$ and $h(t) = Acos$ where $t$ represents time (independent variable), $h$ represents height (dependent variable), $A$ represents the amplitude, and $B$ represents the period of the functions. Task 2 (Algebraic Representation of the Sine and Cosine Functions Considering the Amplitude, Period, Variation, and Phase Shift) draws upon Task 1, in which students are expected to create a more general model, including the phase shift. The expected expressions are $h(t) = A\sin(B\ t) + C$ and $h(t) = A\cos(B\ t) + C$, with $C$ representing the phase shift.

In these tasks, students will use conventional procedures and notations inherent to mathematics while also generalizing various representations and models, extending beyond the context of the problem situation.

## 7. Results and Conclusions

This chapter presents the results obtained during the execution of the tasks by the students participating in the research. For the analysis, the most relevant information was considered based on the analysis categories described in the methodology. These categories are deemed suitable for understanding the process of mathematical modeling undertaken by the students while working with specific trigonometric functions, namely sine and cosine, within a realistic context.

The analysis of the models reinvented by the students during the task execution is conducted on the initial assumptions discussed in the predictive analysis. This approach determines the level of understanding (situational, referential, general, or formal) at which the students are situated. In addition, it seeks to characterize the process of transitioning from informal to formal knowledge. Besides contributing to teaching and learning the sine and cosine trigonometric functions, the primary intention is to comprehend aspects related to the process of mathematization. This includes how students can recognize embedded mathematics within their context, allowing them to describe the behavior of various real-world phenomena. In other words, it enables them to make mathematics more human, as proposed by Freudenthal.

In this context, the results of each situation are categorized by the question, student responses, number of the students, and corresponding percentages.

*Q1:* What does the term "high tide" mean?

*Q2:* What does the term "low tide" mean?

*Q3:* What causes the types of tides?

*Q4:* How often does the tide process repeat?

*Q5:* According to the reading, how often do high and low tides occur?

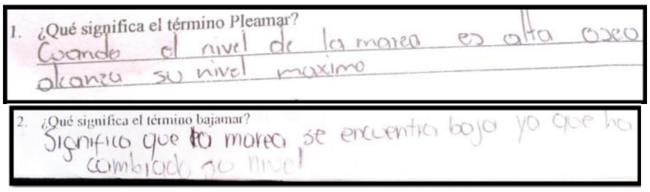
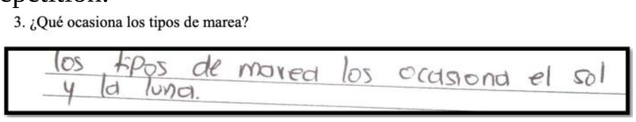



Fig. 2 Evidence of the answers to Q1 and Q2

For Questions 3, 4, and 5, the students provide a detailed characterization of the causes of different types of tides and their repetition intervals. Regarding Question 3, 80% of the students responded in line with the information presented in the reading, explaining the causes of tide types (high and low tides). However, 20% of the students do not give an accurate response to the question, suggesting difficulty in understanding the question.

Concerning Question 5, 60% of the students correctly interpreted that the tide cycle repeats every 24 h based on the reading. Nonetheless, 40% refer not to the overall tide cycle but rather to the period between high and low tides. This aspect is fully addressed (100%) in question 6, where there is explicit mention of the time interval for high and low tides, recognizing this pattern as a periodic phenomenon due to its repetition.



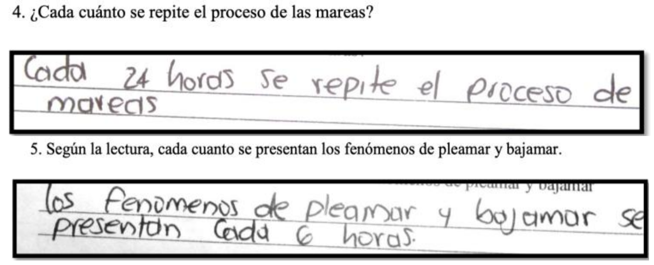



Fig. 3 Evidence of the answers to Q3, Q4, and Q5

## 7.1. Prospective Analysis

*Situation 1: Recognizing Periodic and Variation Phenomena*

This situation was carried out during the first section of the application. Initially, the students watched a video about tides, which provided an overview of what tides are, the types of tides that can occur, and how they manifest. The video was presented to contextualize the students and enhance their understanding of the context of the situation. Subsequently, a discussion was initiated on the basis of the observations from the video, aiming to engage and involve the students in addressing the problem situation. Afterward, the students were provided with a document containing the situation they were to work on individually.

### *7.1.1. Results and Analysis of Situation 1*

*Tidal Behavior (Contextualization)*

Tides are the rises and falls of oceanic waters, including those of open seas, gulfs, and bays. These movements are due to the gravitational attraction of the Moon and the Sun to the water and the Earth. This gravitational force exerted by the Sun and the Moon on the masses of water on Earth causes a rhythmic oscillation of these masses of water due to the Earth’s orbit around the Sun and the Moon’s orbit around the Earth. Thus, there are tides caused by both the Sun and the Moon.

It is a common and easily observable phenomenon. For instance, a boat floats near the shore of the sea for a while, and after a few hours, it rests on the sand. When this happens, it is said that the tide has gone down, and when the opposite occurs, it is said that the tide is high.

There are four stages in the tide process:

1. The sea level gradually rises over several hours.
2. The water level reaches its highest point.
3. The water level gradually decreases over several hours.
4. The water stops descending and reaches its lowest level.

This cycle repeats every 24 h, resulting in high and low tides every 6 h.

### *7.1.2. Results and Analysis of Situation 2*

Taking into account the students’ responses regarding Questions 1 and 2 of Situation 1 - Task 1, it can be said that, in general terms, the students understood the reading presented to them concerning the behavior of tides. They provided accurate definitions of the terms "high tide" (the highest level reached by the tide) and "low tide" (the lowest level the tide can reach), even though they used different terms. In this sense, the reading and presented video provide a context for the situation because this tidal phenomenon is closely related to the student’s context as they live in a city where this behavior is frequently observed. Consequently, this will be the starting point for the students’ mathematical activity, allowing the recognition and characterization of periodic variation phenomena without a mathematical representation but rather from a verbal representation.

### *7.1.3. Results and Analysis of Situation 3*

To develop this situation, the students used the Geogebra program, which presented them with simulations of problem situations (movement of a bicycle wheel and behavior of a spring). Through visualization and animation, the students were able to observe the behavior of the bicycle wheel and the spring, enabling them to answer each question in each task. The program also allowed the students to represent the required tasks in tabular and graphical form, followed by an analysis of their representations.

*Contextualization:* The movement performed by a bicycle wheel is a type of cyclic motion, meaning that it repeats regularly in the same manner.

When moving the bicycle wheel at a constant speed, all cycles of the wheel have the same duration because it takes the same time to complete each revolution. All cyclic movements that repeat at equal time intervals, such as the bicycle’s case, are called periodic motions [35].

*Guiding problem:* Math teacher N. 1 told her that the movement of a bicycle wheel could be represented by a mathematical model. The teacher wants to verify this statement by considering the information provided by her teacher. A bicycle has a fixed marker on the tire of its rear wheel, as shown in the image. The faster the wheel spins, the faster its oscillation.

Regarding the generalization through mathematical models that describe the behavior of a periodic phenomenon, in this case, the Ferris wheel, using strategies that are part of mathematics itself, the students faced difficulties. While they generally managed to characterize the situation in terms of the pattern recognition, regularities, and even tabular and graphical representation, they struggled when it came to mathematizing the situation, i.e., representing it through a mathematical model. This difficulty was evident as only 10% of the students, not independently but with guided reinvention, as proposed in RME, could mathematically represent the problem situation. This reiterates the importance of working in a classroom with mathematical models. Students should not be presented with finished mathematics; they should be allowed to reinvent mathematical objects. Modeling should be considered a fundamental element in achieving this goal.

Considering the level of difficulty that the students faced in the development of Situation 3, due to the relative nature of the concept of a realistic context within the RME framework, it can be concluded that, despite using the Geogebra program that allowed the students to visualize the problem situation and characterize it, they did not reach a formal level of vertical comprehension in mathematical modeling. They did not even reach a general level of comprehension, which suggests the need to work on mathematics in the classroom from a realistic context. This approach would enable students to represent the behavior of phenomena in their context using mathematics, ranging from the simplest to the most complex. In essence, it is about guiding students from an informal understanding to a formal one.

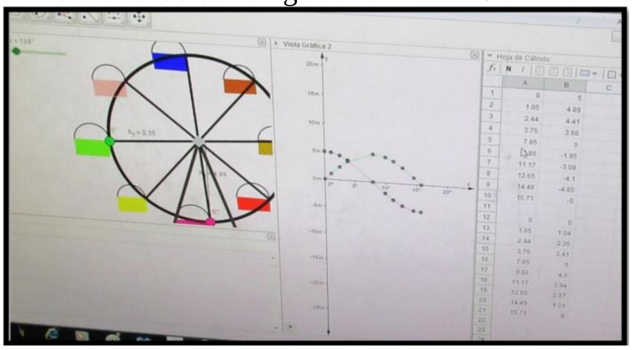


Fig. 4 Screenshot of a student working on a wheel (Elaborated by the authors)

## 7.2. Conclusions

In this section, conclusions and reflections on the teaching and learning process of trigonometric functions, specifically sine and cosine, within a realistic context are presented. These conclusions and reflections arise after implementing a classroom proposal with tenth-grade students from the San Vicente Educational Institution in the Buenaventura District. They are drawn in contrast to the objectives, theoretical framework, and analysis of each of the students' productions.

Consistently with the theoretical and methodological approach of RME, one of the main achievements of this work was characterizing how students, through progressing through different levels of horizontal and vertical mathematization, used various strategies to mathematize a situation in a realistic context. This ranged from using simple notations, where their previous informal knowledge played a crucial role in responding to the situation presented, to using notations specific to the mathematics context, albeit without reaching a general level of mathematical representation. This approach favored the transition not from informal to formal knowledge but from informal to pre-formal knowledge.

In addition, it is pertinent to clarify how each of the specific objectives was achieved during the work:

- The RME offers a theoretical and methodological framework that could be valuable in promoting a unique approach to mathematical modeling processes in the classroom. This could materialize in intervention strategies within mathematics classrooms through the design and implementation of problem situations in a realistic context. For this particular case, periodic natural phenomena that can be modeled using the sine and cosine functions were considered.

- The curriculum references, both the Guidelines (1998) and Standards (2006), propose modeling as one of the five processes of mathematical activity for developing mathematical thinking. In conjunction with the educational perspective proposed by the RME, it can be said that both approaches are closely related as they encourage educators to teach more humanistic mathematics. This allows students to be active participants in constructing their knowledge by reinventing mathematical objects based on their own experiences. Moreover, trigonometry, especially the sine and cosine functions due to their periodic nature, was essential for achieving this goal. This agrees with one of the mathematics standards for secondary education outlined by the Ministry of Education (MEN), using trigonometric functions to model phenomena [7].

- The disciplinary reference is essential for the classroom proposal, providing fundamental elements for students to search for mathematical models to represent a realistic context. This allows them to work with different representations, such as tabular, graphical, and algebraic forms, using symbolic and formal language intrinsic to mathematics. By searching for patterns and regularities in the problem situations, students engage in theoretical elements to analyze their productions from a disciplinary perspective.

Moving on to the second objective related to characterizing the activities' productions carried out by the students, showcasing the process of mathematization concerning theoretical and methodological elements proposed by RME, in conjunction with the use of Geogebra, it can be concluded that:

- The students identified periodic variation patterns with some difficulty from tabular records based on situations from their environment (tide behavior). They employed strategies linked to the context, in which non-mathematical knowledge played a fundamental role in bringing forth models that describe variation and periodicity phenomena. This places the students at a level of horizontal mathematization framed within situational comprehension.

- The students managed to graphically represent a periodic variation function from tabular records linked to specific situations (bicycle wheel behavior and spring movement). They recognized the function's period as the time it takes to complete one cycle of motion and identified the dependence between variables in a functional relationship. Recognizing regularities through notations and mathematical symbols allowed the students to establish mathematical models in an algebraic notation that represented a specific situation. This facilitated the transition from a situational level of comprehension to a referential level.

- In the process of generalizing the sine and cosine trigonometric functions, the students recognized these two functions as periodic functions capable of modeling everyday phenomena with periodic behavior. Using conventional procedures, the students could identify some characteristics of these functions, such as the amplitude, period, and vertical displacement. They also easily associated the curves generated from the tabular representations for each function (sine or cosine). However, when it came to establishing models that incorporate general elements of the sine and cosine functions, the students encountered significant difficulties. This prevented them from reaching a formal level of vertical mathematization, implying that they did not surpass the general level of comprehension.
- Geogebra software acted as an educational tool that enhanced visualization and facilitated the understanding of variation and change phenomena. This was particularly evident in Situations 2 and 3, where the students used the program to observe the behavior of each situation, which supported the efficient development of the proposed tasks. The animation feature enabled the students to observe the behavior of the phenomena in greater detail.
- The process of mathematization developed with the students through the three proposed situations not only highlighted the various strategies used by the students to solve problem situations and advance through different levels of mathematization but also allowed them to recognize mathematics as a means of organizing reality. This approach fostered critical thinking through the interrelationship between daily life and this discipline.

In conclusion, this study demonstrated that the RME framework, coupled with the use of the Geogebra software, can enhance the teaching and learning of trigonometric functions within a realistic context. The study demonstrated that students can progress from recognizing patterns and regularities in real-world phenomena to establishing mathematical models that describe these phenomena using algebraic representations. However, the transition from informal understanding to a formal level of mathematical representation can be challenging, especially when dealing with more generalized mathematical expressions.

The study also highlighted the importance of contextualizing mathematical concepts within students' everyday experiences. Realistic contexts provide a meaningful foundation for students to engage with mathematical content, fostering a deeper understanding of mathematical concepts. This aligns well with the curriculum references that emphasize the importance of mathematical modeling in education.

Furthermore, the role of technology, such as Geogebra, in the visualization and dynamic exploration of mathematical concepts was evident. Technology tools can facilitate comprehension and bridge the concrete world with abstract mathematical representations.

In summary, the study concludes that the RME framework, combined with technology and realistic contexts, can enhance students' understanding of trigonometric functions. The journey from informal understanding to formal mathematical representation requires careful pedagogical planning and scaffolding, but it ultimately offers students a deeper and more meaningful grasp of mathematical concepts that are often perceived as abstract or disconnected from their lives.

## 8. Discussion and Reflection

In alignment with the target of this investigation, which raises reflections regarding the advantages and limitations in teaching the sine and cosine trigonometric functions, the following can be stated:

- Teaching trigonometric functions through the extension of trigonometric ratios, the relationships between sides and angles in the right triangle employing the unit circle, and subsequently straightforwardly constructing their graphs, without establishing a connection with the student's context [3], divests them of the uses and meanings that give rise to the trigonometric function, such as periodic phenomena that facilitate the mathematical modeling or production of models by the student.
- Teaching trigonometric functions from a realistic approach enables the establishment of connections between the concrete (student's everyday life) and the abstract (disciplinary), allowing students to access mathematical knowledge based on their social and natural reality, thus enhancing their learning as it provides them with the opportunity to work at various levels of conceptualization, employ their common sense, mobilize and engage their prior informal knowledge during the resolution processes that arise in their mathematical activity.

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