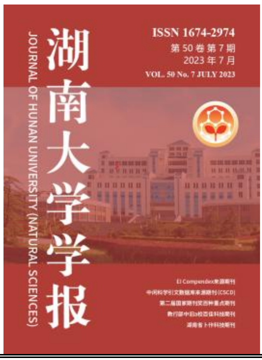

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Solution to the Klein–Gordon Equation Using FEM

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Abstract: The main objective of this paper is to present an approximate numerical solution of the Klein-Gordon equation. For this purpose, the basic concepts associated with the numerical method used to obtain the approximate solution will be presented initially. In our case, we considered as a hypothesis the fact that using the finite element method and in particular the Galerkin method, it is possible to perform a superb numerical approximation of the solution of the Klein-Gordon equation. For this purpose, we will present some of the solutions that have been obtained by employing the modified Adomian decomposition method, via a transformation and Exp-function method and comparison with Adomian's, and using the variational method. This will allow better understanding of the effectiveness of the finite element method because the solutions will be represented in space and time. The novelty results from the fact of being able to take the solutions known as initial and boundary conditions and observe that effectively at great lengths the solution stabilizes at zero and that over time the approximate solutions stabilize. Finally, the simulations obtained by the finite element method indicate that the Klein-Gordon equation is damped nonlinear, its solution stabilizes at -1 and 1, and for short times, the function remains close to zero.

Keywords: approximate solution, finite element method, nonlinear equation, the Klein-Gordon equation, the Galerkin solution.

使用有限元法求解克莱因-戈登方程

摘要：本文的主要目的是提出克莱因-戈登方程的近似数值解。为此，首先将介绍与用于获得近似解的数值方法相关的基本概念。在我们的案例中，我们假设使用有限元方法，特别是伽辽金方法，可以对克莱因-戈登方程的解进行出色的数值逼近。为此，我们将介绍一些通过采用改进的亚多米安分解方法、通过变换和经验值函数方法并与亚多米安的比较以及使用变分方法获得的解。这将有助于更好地理解有限元方法的有效性，因为解将在空间和时间中表示。新颖性源于以下事实：能够采用称为初始条件和边界条件的解，并观察到解在很长的时间内有效地稳定在零，并且随着时间的推移，近似解也稳定下来。最后，有限元方法的模拟表明，克莱因-戈登方程是阻尼非线性的，其解稳定在-1和1，并且在短时间内，函数保持接近于零。

关键词：近似解、有限元法、非线性方程、克莱因-戈登方程、伽辽金解。

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Introduction

The Klein-Gordon (K-G) equation [1] is a type of non-linear hyperbolic equations with partial derivatives. This equation is of great importance in fluid dynamics and solid-state physics. In addition, the Klein-Gordon mathematical model is considered an essential model of quantum mechanics and applied in collision plasma for solution interaction and nonlinear wave equation [2]. We will make an approximation to a set of solutions of the K-G equation using the Galerkin finite element method (FEM) [3].

The K-G equation, Equation (1), is given by

$$\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}=\varphi \quad x\in\mathbb{R},\ t>0 \tag{1}$$

where $\varphi=\varphi(x,t)$ is a function that depends on space and time. There are several methods to obtain the approximate solutions of the Klein-Gordon and nonlinear equations, including the variational iteration method [4-6], homotopy perturbation method [7], the Adomian decomposition method [8], and exponential function method [9].

This article aims to show an approximation to the solution of Equation (1) by the FEM discretizing this equation, both for space and time, obtaining a matrix equation that approximates the model, so the solutions will be a set of linear functions that will depend on the space for each time.

1. Justification

This method allows us to obtain an approximation of the solution of a PDE by dividing a domain $D\in R^n$ into known geometric figures (such as triangles) that complete the original domain, including the boundary conditions as integrals, using Galerkin's residual theory, thus guaranteeing that the construction procedure is completely independent of the boundary conditions in each problem.

The general idea of the n_i FEM is to multiply the governing equation, which for us is (1), by an interpolation function n_i with $i=1,2,\ldots,N+1$ and integrate over the domain of the solution to find the projection equations (Equation (2)).

$$\int_D n_i\left(\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}-\varphi\right)dx=0. \tag{2}$$

The need to discretize the function $\varphi(x,t)$ is evident in such a way that it only depends on the position. This procedure allows the derivatives to decrease their order, using integration by parts, and form a system of linear differential equations, which, according to the initial conditions, generates a tridiagonal stiffness matrix, which can be solved by the Gauss-Jordan method or the Thomas algorithm to solve a tridiagonal system [10]. When comparing the finite difference method with the FEM, it presents greater power in the application of the boundary conditions of the problem. Different physical problems present boundary conditions where derivatives and irregular domains intervene. These boundary conditions are made difficult by finite divided differences because each boundary condition implies a derivative that must be approximated by a quotient of differences in those of the element e_i (domain discretized) and in an irregular edge shape, making the approximations indeterminate [11, 12].

2. Application of the FEM to the Solution of the Klein-Gordon Equation

Take the Sine-Gordon (S-G) equation. The S-G equation is a nonlinear hyperbolic partial differential equation involving the D'Alembert operator [13, 14] and the sine of the unknown function (Equation (3)).

$$\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}=\sin\varphi \quad x\in\mathbb{R},\ t>0, \tag{3}$$

If we consider small oscillations for φ, that is, $\sin\varphi=\varphi$, it implies finding the K-G equation (1). We can pose the initial value problem (IPV) for the Klein-Gordon equation (Equation (4)).

$$\begin{aligned}&\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}=\varphi,\\ &\varphi(0,t)=\varphi_0(t), \qquad \varphi_t(x,0)=g(x),\\ &\varphi(x,0)=f(x), \qquad \varphi_x(0,t)=0,\\ &\qquad\qquad\qquad\qquad \varphi_x(L,0)=p(t).\end{aligned} \tag{4}$$

The discretization and interpolation are indicated in Fig. 1, where $x_1=0$ and $x_{N+1}=L$.

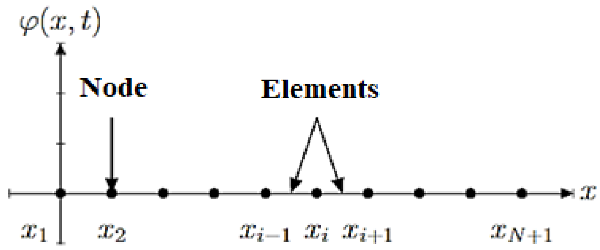

Fig. 1 Discretization of space (Developed by the authors)

The semi-discrete interpolation [15] has the following form (Equation (5)):

$$\varphi(x,t)=\sum_{i=1}^{N+1}\varphi_i(t)n_i(x). \tag{5}$$

with the interpolation function $n_i(x)$, which can be linear, quadratic, or higher-order if desired. In this study, the indicated elements are linear interpolators [15].

Next, we present the elemental formulation. Let the PVI be (4). To weaken the K-G equation, we multiply by a test function $n_k(x) = n_k$ that satisfies the necessary boundary conditions, i.e., Equation (6),

$$\int_0^L n_k(\varphi_{2t} - \varphi_{2x} + \varphi)dx = 0, \tag{3}$$

obtaining Equation (7):

$$\int_0^L (n_k\varphi_{2t} - n_k\varphi_{2x} + n_k\varphi)dx = 0, \tag{4}$$

where the solution of these integrals is given by Equation (8).

$$\begin{aligned}
\int_0^L (n_k\varphi_{2t})dx &= \int_0^L (n_k\varphi_{2t})dx\,, \\
\int_0^L (n_k\varphi_{2x})dx &= n_k\varphi_x|_0^L - \int_0^L (n'_k\varphi_x)dx \\
&= n_k(L)P(t) - \int_0^L n'_k\varphi_x\,dx \\
\int_0^L n_k\varphi\,dx &= \int_0^L n_k\varphi\,dx\,.
\end{aligned} \tag{5}$$

Thus, the integral of the equation is presented in a weak form (Equation (9)).

$$\begin{aligned}
&\int_0^L n_k\varphi_{2t}\,dx - n_k(L)P(t) + \int_0^L n'_k\varphi_x\,dx \\
&\qquad + \int_0^L n_k\varphi\,dx = 0, \\
&-n_k(L)P(t) + \int_0^L (n_k\varphi_{2t} + n'_k\varphi_x + n_k\varphi)\,dx \\
&\qquad = 0
\end{aligned} \tag{9}$$

By [3], as $n_k = n_k(x)$ with $k = 1,2,\dots,N+1$ and derived Equation (5) with respect to x, we have Equation (10).

$$\begin{aligned}
-\delta_{k,N+1}P(t) + \int_0^L \Bigg(&\sum_{i=1}^{N+1} n_k\,n_i(\varphi_{2t})_i \\
&+ \sum_{i=1}^{N+1} n'_k n'_i\varphi_i \\
&+ \sum_{i=1}^{N+1} n_k\,n_i\varphi_i\Bigg)dx = 0,
\end{aligned} \tag{10}$$

like $\varphi(i,t) = \varphi(t)$ and $\varphi_{2t}(i,t) = \varphi_{2t}(t)$. If we set $A_{ki} = \int_0^L n_k\,n_i\,dx$ and $B_{ki} = \int_0^L n'_k n'_i\,dx$, by [3], $A = \sum_{\mathrm{e}} M_G$ with $m_e = \int_{x_i}^{x_{i+1}} \mathbb{N}\,p\mathbb{N}^T\,dx$, and p is a start parameter; furthermore, $B = \sum_{\mathrm{e}} K_G$ with $k_{ei} = \int_{x_i}^{x_{i+1}} \left(\mathbb{N}'p\mathbb{N}'^T + \mathbb{N}q\mathbb{N}^T\right)dx$ and $\mathbb{F}_k = \delta_{k,N+1}P(t)$.

$$N = \begin{bmatrix} \dfrac{x_{i+1} - x}{x_{i+1} - x_i} \\ \dfrac{x - x_i}{x_{i+1} - x_i} \end{bmatrix}, \qquad N' = \begin{bmatrix} \dfrac{-1}{x_{i+1} - x_i} \\ \dfrac{1}{x_{i+1} - x_i} \end{bmatrix}$$

We find writing for φ as a vector; therefore, we have Equation (11).

$$\sum_{i=1}^{N+1} (A_{ki}\ddot{\varphi}_i + (A_{ki} + B_{ki})\varphi_i) = \mathbb{F}_k(t). \tag{11}$$

If we make the following substitutions (Equation (12)),

$$\begin{aligned}
\mathbb{A} &= \sum_{\mathrm{i}} A_{ki} = \sum_{\mathrm{i}} M_G, \quad \text{with} \quad M_G = NN^T \\
\mathbb{B} &= \sum_{\mathrm{i}} B_{ki} = \sum_{\mathrm{i}} K_G, \quad \text{with} \quad K_G = N'N'^T
\end{aligned}. \tag{12}$$

our initial value problem equation (4) can be written as PVI in vector form (Equation (13)).

$$\mathbb{A}\ddot{\varphi}_i + (\mathbb{A} + \mathbb{B})\varphi_i = \mathbb{F}, \tag{13}$$

with the conditions $\begin{array}{ccc} \varphi(0) & = & \varphi_0 \\ \dot{\varphi}(0) & = & \dot{\varphi}_0 \end{array}$.

The vectors $\varphi_0, \dot{\varphi}_0$ correspond to the discretization of the initial conditions f, g; in general, they are taken as the respective values of $f(x), g(x)$ in the nodes, that is, Equation (14).

$$\begin{array}{ccccc} \varphi(0) & = & \varphi_0 & = & [f(0)\ f(x_2)\ f(x_3)\ \cdots\ f(x_N)\ L] \\ \dot{\varphi}(0) & = & \dot{\varphi}_0 & = & [g(0)\ g(x_2)\ g(x_3)\ \cdots\ g(x_N)\ L] \end{array} \tag{14}$$

Thus, the system takes the following form (Equation (15)).

$$\begin{aligned}
a_{11}\ddot{\varphi}_1 + a_{12}\ddot{\varphi}_2 + \cdots + b_{11}\varphi_1 + \cdots &= F_1, \\
a_{j1}\ddot{\varphi}_1 + a_{22}\ddot{\varphi}_2 + \cdots + b_{21}\varphi_1 + \cdots &= F_2, \\
&\vdots \\
a_{N+1,1}\ddot{\varphi}_1 + a_{N+1,2}\ddot{\varphi}_2 + \cdots + b_{N+1,1}\varphi_1 + \cdots & \\
= F_{N+1}.
\end{aligned} \tag{15}$$

This system that we built does not consider all the conditions of the problem; therefore, it must include the condition $\varphi(0,t) = \varphi_0(t)$ for all $t \geq 0$; we have Equation (16):

$$\begin{aligned}
\varphi_1 &= \varphi_0(t), \\
a_{21}\ddot{\varphi}_1 + a_{22}\ddot{\varphi}_2 + \cdots + b_{21}\varphi_1 + \cdots &= F_2, \\
&\vdots \\
a_{j1}\ddot{\varphi}_1 + a_{j2}\ddot{\varphi}_2 + \cdots + b_{j1}\varphi_1 + \cdots &= F_j, \\
&\vdots \\
a_{N+1,1}\ddot{\varphi}_1 + a_{N+1,2}\ddot{\varphi}_2 + \cdots + b_{N+1,1}\varphi_1 + \cdots & \\
= F_{N+1},
\end{aligned} \tag{16}$$

which we can write as Equation (17).

$$\begin{aligned}
\varphi_1 &= \varphi_0(t), \\
a_{22}\ddot{\varphi}_2 + \cdots + b_{22}\varphi_2 + \cdots &= F_2 - a_{21}\ddot{\varphi}_0 - b_{21}\varphi_0, \\
&\vdots \\
a_{j2}\ddot{\varphi}_2 + \cdots + b_{j2}\varphi_2 + \cdots &= F_j - a_{j1}\ddot{\varphi}_0 - b_{j0}\varphi_0, \\
&\vdots \\
a_{N+1,2}\ddot{\varphi}_2 + \cdots + b_{N+1,2}\varphi_2 + \cdots & \\
= F_{N+1} - a_{N+1,1}\ddot{\varphi}_0 - b_{N+1,1}\varphi_0.
\end{aligned} \tag{6}$$

It is possible to demonstrate the effect of the conditions in the system, allowing us to rewrite the system as follows (Equation (18)):

$$\mathbb{M}\ddot{\varphi}_i + (\mathbb{M} + \mathbb{K})\varphi_i = \mathbb{F}. \tag{18}$$

with initial conditions $\begin{array}{ccc} \varphi(0) & = & \varphi_0 \\ \dot{\varphi}(0) & = & \dot{\varphi}_0 \end{array}$.

The second-order system that we have has the characteristic that $\varphi = \varphi(t)$ and $\mathbb{F} = \mathbb{F}(t)$; hence the discretization in the time variable (Fig. 2).

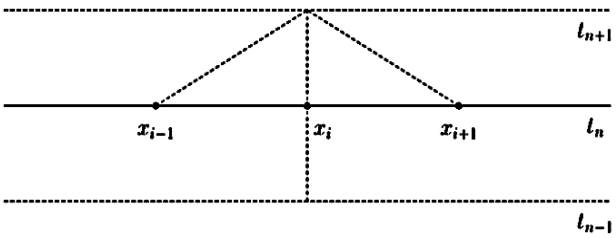

Fig. 2 Space-time discretization (Developed by the authors)

We have $\Delta t = t_n - t_{n-1} = t_{n+1} - t_n$ to make $\varphi(t) = \varphi(t_n) = \varphi_n$ y $\mathbb{F}(t) = \mathbb{F}(t_n) = \mathbb{F}_n$ using the centered finite difference method [16, 17] (Fig. 3).

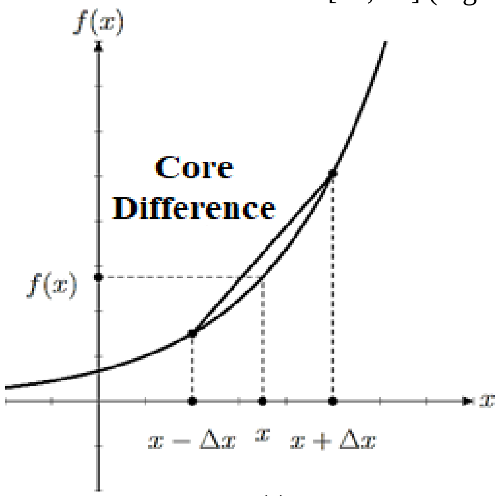

Fig. 3 Centered difference for f(x) (Developed by the authors)

For the curve $f(x)$, we have Equation (19).

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{x+\Delta x - x + \Delta x} \qquad (19)$$
$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

For the case of f'', see Fig. 4.

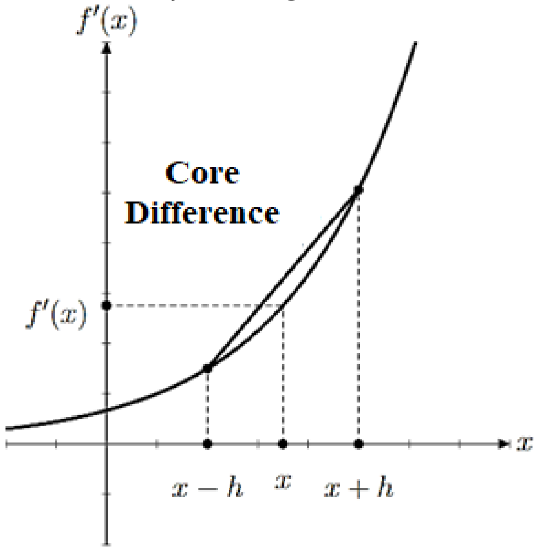

Fig. 4 Centered difference for f′(x) (Developed by the authors)

So for the curve $f'(x)$, we have Equation (20).

$$\begin{aligned} f''(x) &= \frac{f'(x+h) - f(x-h)}{2h} \\ &= \frac{\frac{f(x+2h) - f(x)}{2h} - \frac{f(x) - f(x-2h)}{2h}}{2h}. \\ &= \frac{f(x+2h) - 2f(x) + f(x-2h)}{(2h)^2}. \end{aligned} \qquad (20)$$

Let $2h = \Delta x$ Δx. Then, we have Equation (21).

$$f''^{(x)} = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2} \qquad (21)$$

Using the centered differences [18] for the discretization of the variable φ in time, we have Equations (22) and (23).

$$\dot{\varphi}_n = \frac{\varphi_{n+1} - \varphi_{n-1}}{2\Delta t}. \qquad (22)$$
$$\ddot{\varphi}_n = \frac{\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}}{(\Delta t)^2} \qquad (23)$$

Substituting into the governing Equation (18), we have Equation (24):

$$\mathbb{M}\frac{\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}}{(\Delta t)^2} + (\mathbb{M} + \mathbb{K})\varphi_n = \mathbb{F}_n, \qquad (24)$$

which we can write as Equation (25):

$$\begin{aligned} \mathbb{M}\varphi_{n+1} &= \mathbb{M}(2\varphi_n - \varphi_{n-1}) - (\mathbb{M} + \mathbb{K})\varphi_n(\Delta t)^2 + \mathbb{F}_n(\Delta t)^2 \\ \mathbb{M}\varphi_{n+1} &= (2\mathbb{M} - (\mathbb{M} + \mathbb{K})(\Delta t)^2)\varphi_n - \mathbb{M}\varphi_{n-1} + (\Delta t)^2\mathbb{F}_n \end{aligned} \qquad (25)$$

3. Analysis and Results

In this section, we will take solutions of Equation (1) that are theoretical references of great importance and have been developed using methods other than the FEM proposed in different research articles. These solutions will be the initial and boundary conditions of our problem.

1. [8] proposes a solution of Equation (26):

$$u(x,t) = x\cos(t). \qquad (26)$$

By taking this solution in our problem as the initial and boundary conditions, the solutions in space (Fig. 5) and time (Fig. 6) were obtained through the FEM.

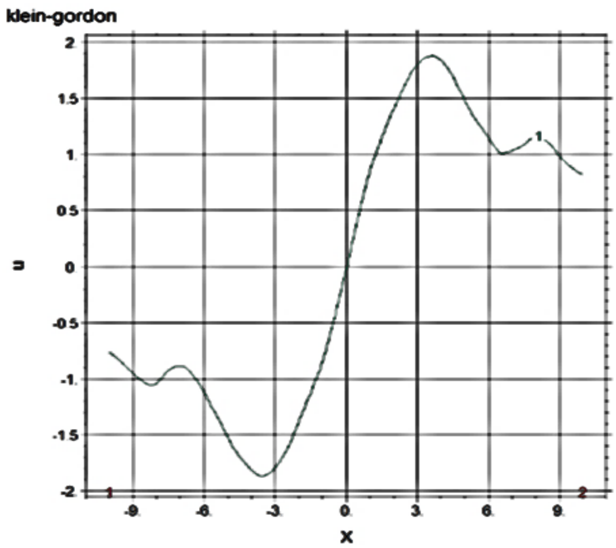

Fig. 5 Space, u(x, t) (Developed by the authors)

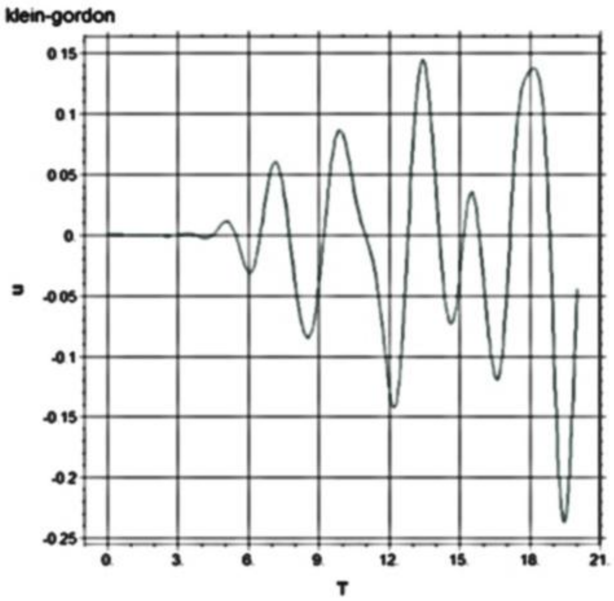

Fig. 6 Time, u(x, t) (Developed by the authors)

Fig. 5 allows us to visualize that the solution stabilizes at -1 and 1. Fig. 6 shows that for short times, the function remains close to zero.

2. [9] presents Equations (27), (28), and (29) as three different solutions.

$$u(x,t) = sech^2(x-t). \tag{27}$$

$$u(x,t) = \mathrm{sech}(x-t). \tag{28}$$

$$u(x,t) = cosh^{1/2}\big(2(x-t)\big), \tag{29}$$

Examining each of them, by substituting them in our problem as the initial and boundary conditions, the solutions in space (Fig. 7) and time (Fig. 8) were obtained through the FEM for Equation (27); in space (Fig. 9) and time (Fig. 10) - for Equation (28); in space (Fig. 11) and time (Fig. 12) - for Equation (29).

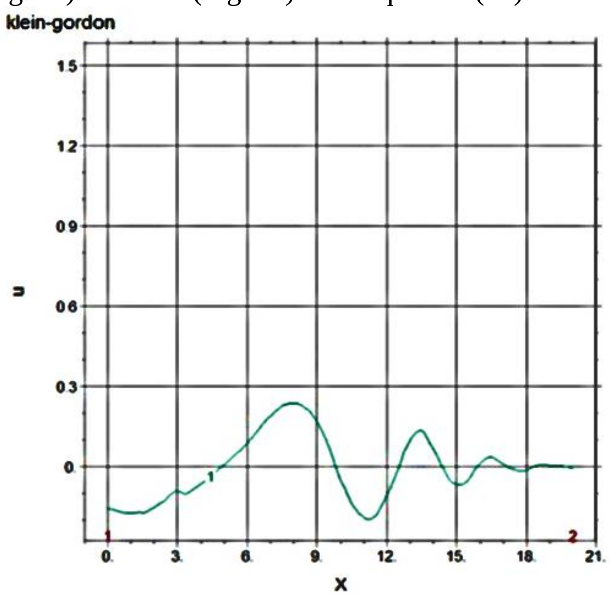

Fig. 7 Space, u(x, t) (Developed by the authors)

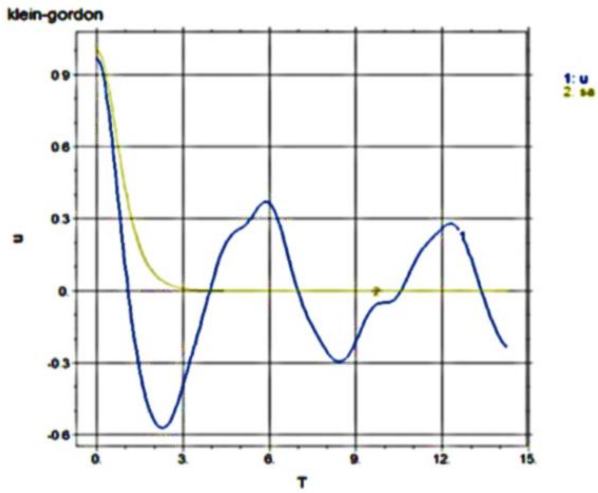

Fig. 8 Time, u(x, t) vs. approximate solution (Developed by the authors)

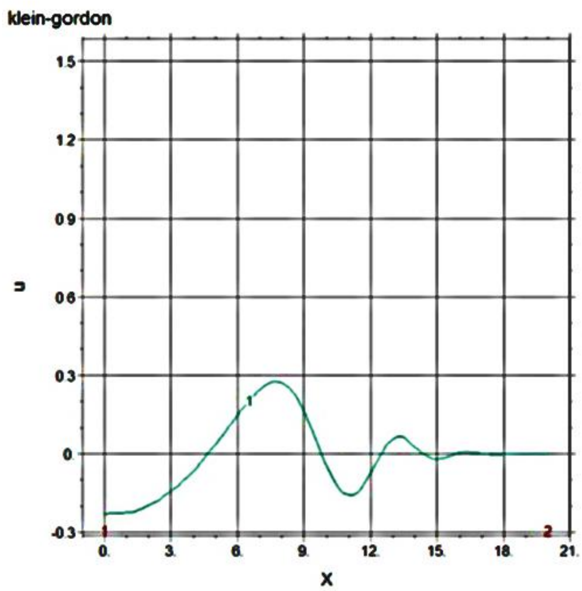

Fig. 9 Space, u(x, t) (Developed by the authors)

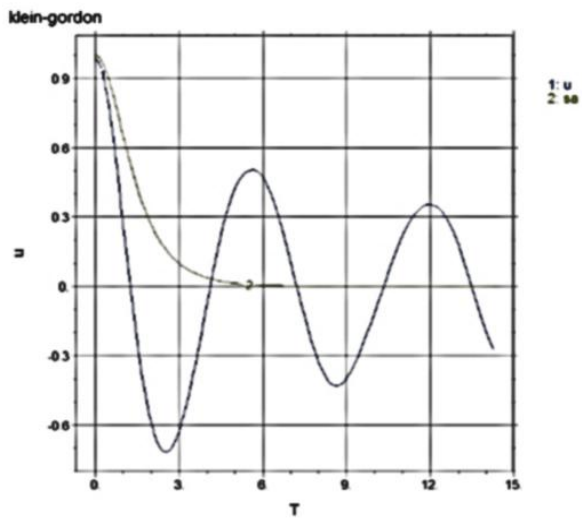

Fig. 10 Time, $u(x,t)$ vs. approximate solution (Developed by the authors)

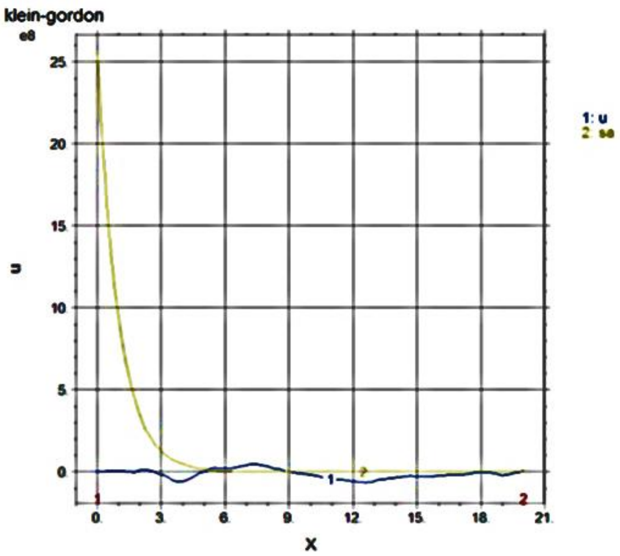

Fig. 11 Space, u(x, t) vs. approximate solution (Developed by the authors)

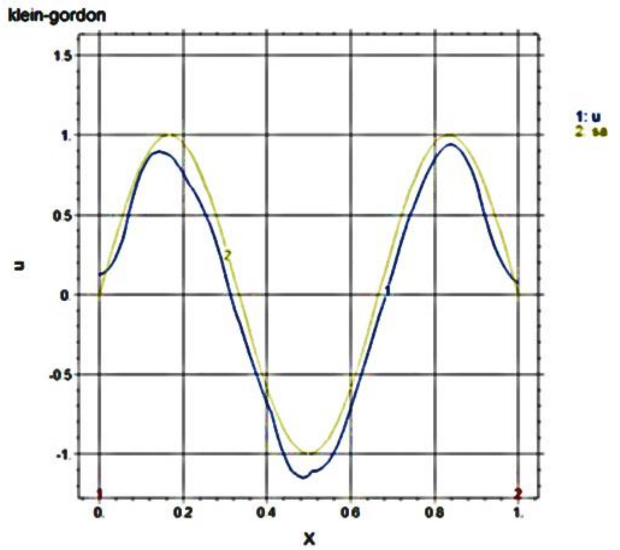

Fig. 12 Space, u(x, t) vs. approximate solution (Developed by the authors)

Fig. 7 shows how the solution stabilizes at zero, which is expected for large lengths. In addition, it is evident in Fig. 8 that the function in time stabilizes against the approximate solution.

Fig. 8 shows the stability at zero, and Fig. 9 presents the same behavior.

Fig. 10 shows that the proposed solution behaves initially far from the found solution, but for large lengths, the solutions approach zero.

3. [6] obtained Equation (30) as a solution.

$$u(x,t) = sen(3\pi x). \tag{30}$$

When used in our problem as an initial condition, we obtained the solutions in space (Fig. 12) and time (Fig. 13).

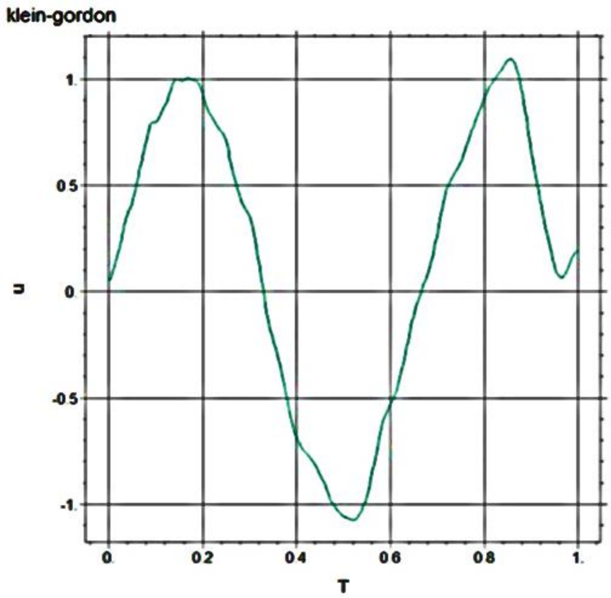

Fig. 13 Time, u(x, t) (Developed by the authors)

The initial condition solution proposed by [6] is a solution used for its 3D graph, and the planes that cut perpendicularly coincide with our solution (Fig. 12).

4. Conclusions

The most relevant findings in this study are shown below:

The numerical solution obtained through the MEF of the Klein-Gordon equation is damped nonlinear, and it was possible to see that the solution stabilizes at -1 and 1 and that for short times, the function remains close to zero.

When using the finite element method to solve the Klein-Gordon equation, it was possible to observe as a novel and innovative result that it is possible to validate other solutions of this equation obtained with other methods, taking them as initial and border conditions, which will allow

The solution stabilizes at zero, which is expected for large lengths. In addition, we showed how the function in time stabilizes against the approximate solution.

It was possible to verify that using the proposed numerical method, an approximate matrix solution to the Klein-Gordon equation was found to be acceptable since when taking other referents that obtained an approximate solution by other methods, they tend to behave the same for long lengths.

The hypothesis that Galerkin's FEM is a crucial mathematical tool when trying to find the approximate solution of a partial differential equation was strengthened.

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