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An Approach to the Stability of Systems of Differential Equations of Fractional Order with Time Delay

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Abstract: In this paper, a stability study is carried out for systems of fractional order differential equations with time delay. These systems of differential equations of fractional order with time delay are of current interest due to their applicability since several models from different areas of knowledge, where the current development depends to a great extent on the history that has elapsed, are faithful reflections of systems of differential equations with time delay. The article initially presents and describes the fundamental concepts of the study of stability of systems of ordinary differential equations of fractional order, both linear and nonlinear, and then these concepts are presented for differential equations with time delay. The results of this research provide fundamental tools in the study of the stability of systems of differential equations of fractional order with time delay that can be applied to a variety of models from the applied sciences.

Keywords: fractional analysis, fractional differential equations, stability of fractional differential equations, fractional differential equations with delay.

时滞分数阶微分方程组稳定性的一种方法

摘要：本文对时滞分数阶微分方程组进行稳定性研究。这些带有时滞的分数阶微分方程组由于其适用性而引起了当前的兴趣，因为来自不同知识领域的几个模型（其中当前的发展在很大程度上取决于已经过去的历史）是系统的忠实反映。具有时滞的微分方程。本文首先提出并描述了分数阶线性和非线性常微分方程系统稳定性研究的基本概念，然后针对时滞微分方程提出了这些概念。这项研究的结果为研究时滞分数阶微分方程组的稳定性提供了基本工具，可应用于应用科学的各种模型。

关键词：分数阶分析、分数阶微分方程、分数阶微分方程的稳定性、时滞分数阶微分方程。

Introduction

Fractional analysis has a long history since its origin. It originated on September 30, 1695, when

L'Hopital addressed a letter to Leibniz arguing a question regarding the notation for the $n-$th derivative of a function $\frac{d^n}{dx^n}$ for the case that $n=\frac{1}{2}$, to which Leibniz replied that his result was "an obvious paradox from which interesting consequences will be obtained in the future" [1]. Fractional calculus has been the field of interest of great mathematicians, such as Euler, who, in 1738, refers to interpolations between integer orders of a derivative and conceives the fractional derivative with the help of interpolation; in 1819, Lacroix following Euler's idea presents an exact formula for the evaluation of $\frac{d^{1/2}}{dx^{1/2}}x^{\alpha}$, and Fourier suggests presenting it as a definition of fractional derivatives:

$$\frac{d^{\alpha}}{dx^{\alpha}}f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lambda^{\alpha}\ d\lambda\int_{-\infty}^{\infty}f(t)\cos\left(\lambda x-t\lambda+\frac{p\pi}{2}\right)\ dt.$$

In 1820, Laplace proposed an argument for the fractional derivative for functions that can be represented by an integral of the type $\int T(t)t^{-x}dt$. The first application was presented by Abel in 1823, when he applied the results of the fractional calculus to the solution of an integral obtained in the formulation of the isochrone problem.

The works presented by the mathematicians motivated Liouville, who published a series of articles between 1832 and 1835 where he presented the first formal definition of the concept of fractional derivative and is considered the father of fractional calculus. In 1847, Riemann wrote an article modifying Liouville's definition of the fractional operator that is known today as the Riemann-Liouville derivative [2].

It is possible to continue in this timeline presenting the most relevant mathematicians in terms of significant contributions on fractional calculus, however, the greatest development of fractional calculus has occurred in the last five decades, which is evident not only in the number of publications presented on the subject but also in the various international congresses devoted exclusively to the subject in question.

The interest in this branch of mathematics was so great that in 1974 Oldham and Spanier, specialists in electrochemistry, presented the first book dedicated exclusively to fractional calculus and its applications, in which basic properties of derivatives and integration operators of fractional type applied in diffusion problems and in classical analysis are collected. Later, with the great applicability of fractional analysis to various areas of knowledge, they became interested in the study of fractional differential equations due to the great interest they have in the modeling of anomalous processes for many branches of applied sciences [3, 4].

One of the first works dedicated to fractional differential equations is that of Miller-Ross in 1993, where an introduction to fractional calculus is given and subsequently the explicit solutions of some fractional differential equations are analyzed, focusing on those linear ones with constant coefficients [5]. In 1999, Podlubny published a monograph focused on providing basic properties of several fractional derivatives and integration operators and on obtaining solutions to fractional differential equations by the Laplace transform [6]. It is also worth mentioning the works published by Kilbas and Trujillo, who in recent decades have focused on the study of fractional differential equations, publishing perhaps one of the first books in Spanish devoted exclusively to fractional differential equations [7].

Within fractional differential equations, one of the fields still under exploration is the qualitative approach focused on the concepts of stability. There is still no established theory of the stability of differential equations; however, in the last two decades, valid contributions were made to this theory. In 1996, Matignon for the first time analyzed the stability of systems of linear differential equations of fractional type of finite dimension applied to control theory; for these systems presented in the form of state space, he presented the so-called internal and external stabilities. He mentioned that the stabilities are guaranteed if the roots of some polynomials are out of $|\arg(\sigma)|\leq\frac{\alpha\pi}{2}$, thus generalizing the known results for the ordinary case $\alpha=1$. Since then, several researchers have been interested in contributing to the study of the stability of linear fractional differential equation systems [8].

For the nonlinear case, the stability analysis is much more complex; however, some authors studied the system of nonlinear differential equations of fractional order from different approaches, where some methods of stability analysis for the ordinary case were extended to the fractional order. For example, in 2007, Tarasov mentions that the concept of stability with respect to fractional variations is broader than in the classical sense of Lyapunov or asymptotic stability. He states that fractional stability includes the concept of classical stability as a special case $\alpha=1$ and that the systems can be unstable with respect to the first variation of states and be stable with respect to the fractional variation [9]. Therefore, fractional derivatives extend the possibility of investigating the properties of differential equation systems.

Later in 2009, Li, Chen, and Podlubny analyzed the stability of systems of nonlinear ordinary differential equations of fractional type using the derivative operator in the Caputo or Riemann-Liouville sense for order $0<\alpha<1$. They managed to extend the concepts of stability of the classical order to systems of differential equations of fractional order, establishing that the generalized energy drop of a dynamic system does not have to be exponential for the system to be stable; this energy decrease can be of any type, including the decrease of the power law [10-12].

Subsequently, Agarwal, Hristova, and O'Regan presented stability concepts focused on Lyapunov-type functions and a type of stability called strict stability for various derivative operators, among which is the derivative in the sense of Riemann-Liouville, Caputo, Hadamard, and Caputo-Dini [13-17].

This paper presents fundamental results of studying the stability of systems of fractional order differential equations, focusing on the stability of systems of fractional differential equations with time delay. This study is presented in four sections, as seen in diagram 1; in the first section, the fundamental preliminaries of fractional operators and their properties are presented; in the second section, the fundamental concepts of fractional order differential equations are presented; in the third section, the concepts of stability of systems of linear, nonlinear, and time-delayed fractional order differential equations are presented; in the last section, the research results are discussed.

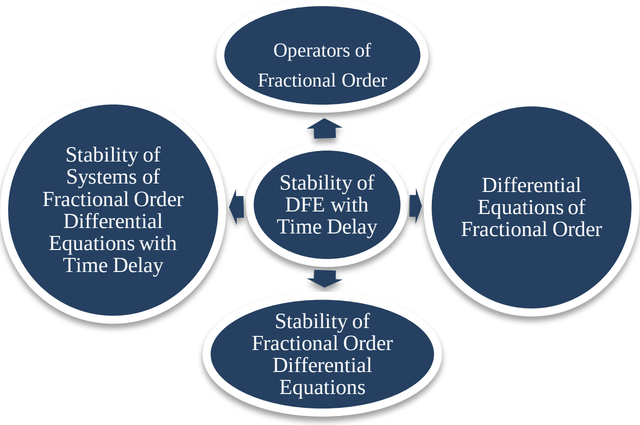

Fig. 1 Methodology of the study (The authors)

1. Operators of Fractional Order

In this section, the fundamental preliminaries of fractional analysis are presented. The definitions and properties of the derivative and integral in the sense of Riemann-Liouville and Caputo are presented [18-22]. In the following definitions $\mathbb{R}e(\alpha)$ denotes the real part of the number α.

Definition 1: Let $f \in L^1[a,b]$ and $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) > 0$. The Riemann-Liouville integrals of order α of the function f are given by

$$\mathcal{J}^{\alpha}_{a^+,x}[f(x)] = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt, x > a \quad (1)$$

and

$$\mathcal{J}^{\alpha}_{b^-,x}[f(x)] = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\,dt, x < b$$

called left and right integrals, respectively.

Definition 2: The Riemann-Liouville derivatives of order α of the function f are given by

$$\begin{aligned}\mathcal{D}^{\alpha}_{a^+,x}[f(x)] &= \frac{d^n}{dx^n}\left(\mathcal{J}^{n-\alpha}_{a^+,x}[f(x)]\right)\\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x \frac{f(t)}{(t-x)^{\alpha+1-n}}\,dt \qquad x > a\end{aligned} \quad (2)$$

and

$$\begin{aligned}\mathcal{D}^{\alpha}_{b^-,x}[f(x)] &= (-1)^n\frac{d^n}{dx^n}\left(\mathcal{J}^{n-\alpha}_{b^-,x}[f(x)]\right)\\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b \frac{f(t)}{(x-t)^{\alpha+1-n}}\,dt, \quad x < b\end{aligned}$$

called the left and right derivatives, respectively, where $n = \begin{cases}\lfloor \mathbb{R}e(\alpha)\rfloor + 1 & if \quad \alpha \notin \mathbb{N}\\ \alpha & if \quad \alpha \in \mathbb{N}\end{cases}$, and $\lfloor\cdot\rfloor$ denotes the integer part of $\cdot$.

Some of the properties satisfied by the fractional Riemann-Liouville operators are presented below. Let $f, g \in L^1[a,b]$, $\alpha, \beta \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) > 0$, $\mathbb{R}e(\beta) > 0$, and $\lambda, \mu \in \mathbb{R}$; then,

1) $\mathcal{J}^{\alpha}_{a^+,x}[\lambda f(x) + \mu f(x)] = \lambda\mathcal{J}^{\alpha}_{a^+,x}[f(x)] + \mu\mathcal{J}^{\alpha}_{a^+,x}[g(x)]$.
2) $\mathcal{D}^{\alpha}_{a^+,x}[\lambda f(x) + \mu f(x)] = \lambda\mathcal{D}^{\alpha}_{a^+,x}[f(x)] + \mu\mathcal{D}^{\alpha}_{a^+,x}[g(x)]$.
3) $\left(\mathcal{J}^{\alpha}_{a^+,x}\mathcal{J}^{\beta}_{a^+,x}\right)[f(x)] = \left(\mathcal{J}^{\beta}_{a^+,x}\mathcal{J}^{\alpha}_{a^+,x}\right)[f(x)] = \mathcal{J}^{\alpha+\beta}_{a^+,x}[f(x)]$.
4) $\left(\mathcal{D}^{\alpha}_{a^+,x}\mathcal{D}^{\beta}_{a^+,x}\right)[f(x)] = \mathcal{D}^{\alpha+\beta}_{a^+,x}[f(x)] - \sum_{j=1}^{n}\mathcal{D}^{\beta-j}_{a^+,x}[f(x)]|_{x=a}\frac{(x-a)^{-j-\alpha}}{\Gamma(1-j-\alpha)}$ c.p.t. in $[a,b]$, with $m, n \in \mathbb{N}$, $m-1 < \alpha < m$, and $n-1 < \beta < n$.
5) $\lim_{\alpha\to 0}\mathcal{J}^{\alpha}_{a^+,x}[f(x)] = \lim_{\alpha\to 0}\mathcal{D}^{\alpha}_{a^+,x}[f(x)] = f(x)$.
6) $\left(\mathcal{D}^{\alpha}_{a^+,x}\mathcal{J}^{\alpha}_{a^+,x}\right)[f(x)] = f(x)$.
7) $\left(\mathcal{J}^{\alpha}_{a^+,x}\mathcal{D}^{\alpha}_{a^+,x}\right)[f(x)] = f(x) - \sum_{j=1}^{n}\mathcal{D}^{\alpha-j}_{a^+,x}[f(x)]|_{x=a}\frac{(x-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}$.
8) $\left(\mathcal{D}^{\beta}_{a^+,x}\mathcal{J}^{\alpha}_{a^+,x}\right)[f(x)] = \mathcal{J}^{\alpha-\beta}_{a^+,x}[f(x)]$ if $\mathbb{R}e(\alpha) \geq \mathbb{R}e(\beta)$.

Caputo's fractional derivative allows considering initial conditions of natural order to mathematical models represented by fractional differential equations. These fractional derivative operators are defined as follows:

Definition 3: Let f be an absolutely continuous function (AC) on $[a,b]$, $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) > 0$ and $n = \begin{cases}\lfloor \mathbb{R}e(\alpha)\rfloor + 1 & if \quad \alpha \notin \mathbb{N}\\ \alpha & if \quad \alpha \in \mathbb{N}\end{cases}$. The Caputo derivatives for the function f are given by

$$\begin{aligned}{}_c\mathfrak{D}^{\alpha}_{a^+,x}[f(x)] &= \mathcal{J}^{n-\alpha}_{a^+,x}[f^{(n)}(x)]\\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}\,dt, x > a\end{aligned}$$

and

$$\begin{aligned}{}_c\mathfrak{D}^{\alpha}_{b^-,x}[f(x)] &= (-1)^n\,\mathcal{J}^{n-\alpha}_{b^-,x}[f^{(n)}(x)]\\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha+1-n}}\,dt, x < b\end{aligned}$$

called left and right derivatives, respectively.

Some properties satisfied by Caputo's fractional operators are presented below. Let $\alpha \in \mathbb{C}$ such that $\mathbb{R}e(\alpha) > 0$, $f \in C[a,b]$, $f^{(n)} \in AC[a,b]$, and $n =$

$\begin{cases}\lfloor \mathbb{R}e(\alpha)\rfloor + 1 & if \;\; \alpha \notin \mathbb{N} \\ \alpha & if \;\; \alpha \in \mathbb{N}\end{cases}$. Then, the following properties are satisfied:

1) ${}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{a^+,x}[f(x)] = \mathcal{D}^{\alpha}_{a^+,x}[f(x)] - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}$ if $x > a$.

2) ${}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{b^-,x}[f(x)] = \mathcal{D}^{\alpha}_{b^-,x}[f(x)] - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha}$ if $x < b$.

3) If $\mathbb{R}e(\alpha) \notin \mathbb{N}$ or $\alpha \in \mathbb{N}$, then,

$\left({}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{a^+,x}\ \mathcal{J}^{\alpha}_{a^+,x}\right)[f(x)] = f(x)$,

$\left({}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{b^-,x}\ \mathcal{J}^{\alpha}_{b^-,x}\right)[f(x)] = f(x)$.

4) If $\mathbb{R}e(\alpha) \in \mathbb{N}$ and $\mathrm{Im}(\alpha) \neq 0$, then,

$$\left({}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{a^+,x}\ \mathcal{J}^{\alpha}_{a^+,x}\right)[f(x)] = f(x) - \frac{1}{\Gamma(n-\alpha)}\mathcal{J}^{\alpha+1-n}_{a^+,x}[f(x)]|_a (x-a)^{n-\alpha},$$

$$\left({}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{b^-,x}\ \mathcal{J}^{\alpha}_{b^-,x}\right)[f(x)] = f(x) - \frac{1}{\Gamma(n-\alpha)}\mathcal{J}^{\alpha+1-n}_{b^-,x}[f(x)]|_b (b-x)^{n-\alpha}.$$

5) If $\mathbb{R}e(\alpha) > 0$ and $f \in \mathcal{C}^n[a,b]$, then,

$$\left(\mathcal{J}^{\alpha}_{a^+,x}\ {}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{a^+,x}\right)[f(x)] = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

$$\left(\mathcal{J}^{\alpha}_{b^-,x}\ {}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{b^-,x}\right)[f(x)] = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!}(b-x)^k.$$

2. Differential Equations of Fractional Order

This section presents the definitions of fractional differential equations using the derivative operators in the sense of Riemann-Liouville and Caputo [20-22]. In addition, the definition of the Mittag-Leffler function and its fundamental properties are presented.

Definition 4: An ordinary differential equation of fractional order α is defined as

$$\mathbb{D}^{\alpha}_{t_0,t}[x] = f\left(t, x, \mathbb{D}^{\alpha_1}_{t_0,t}[x], \mathbb{D}^{\alpha_2}_{t_0,t}[x], \dots, \mathbb{D}^{\alpha_{m-1}}_{t_0,t}[x]\right) \quad (3)$$

where $x(t)$ is a real domain complex unknown function, $f(t, x, x_1, x_2, x_3, \dots, x_{n-1})$ is a known function, and $\mathbb{D}^{\alpha_k}_{t_0,t}$ for $k = 1,2,3,\dots,m-1$ are fractional differential operators such that $0 < \mathbb{R}e(\alpha_1) < Re(\alpha_2) < Re(\alpha_3) < \cdots < Re(\alpha_{m-1}) < Re(\alpha)$ for $m \in \mathbb{Z}_+, m \geq 2$.

Definition 5: An ordinary differential equation of fractional order α of linear type is defined as

$$\mathbb{D}^{\alpha}_{t_0,t}[x] + a_0(t)x + \sum_{k=1}^{m-1} a_k(t)\mathbb{D}^{\alpha_k}_{t_0,t}[x] = f(t) \quad (4)$$

where $x(t)$ is a real domain complex unknown function, $a_k(t)$ for $k = 0,1,2,3,\dots,m-1$ and $f(t)$ are known functions, $\mathbb{D}^{\alpha}_{t_0,t}$ for $k = 1,2,3,\dots,m-1$ are fractional differential operators such that $0 < \mathbb{R}e(\alpha_1) < Re(\alpha_2) < Re(\alpha_3) < \cdots < Re(\alpha_{m-1}) < Re(\alpha)$ for $m \in \mathbb{Z}_+, m \geq 2$.

If the functions $a_k(t)$ are constant for all $k = 0,1,2,3,\dots,m-1$, equation (4) is said to have constant coefficients; on the contrary, if at least one of them is variable, the equation is said to have variable coefficients. Now, if $f(t) = 0$, the linear fractional differential equation of ordinary type is said to be homogeneous.

The Cauchy-type problem for the Riemann-Liouville fractional derivative presented in Equation (1) is given by

$$\begin{cases}\mathcal{D}^{\alpha}_{t_0,t}[x(t)] & = f\left(t, x, \mathcal{D}^{\alpha_1}_{t_0,t}[x], \mathcal{D}^{\alpha_2}_{t_0,t}[x], \dots, \mathcal{D}^{\alpha_{m-1}}_{t_0,t}[x]\right) \\ \mathcal{D}^{\alpha-k}_{t_0,t}[x(t)]|_{t=t_0} & = b_k, \quad b_k \in \mathbb{C}, \quad k = 1,2,3,\dots,m\end{cases} \quad (5)$$

where $m = \begin{cases}\lfloor \mathbb{R}e(\alpha)\rfloor + 1 & if \;\; \alpha \notin \mathbb{Z}_+ \\ \alpha & if \;\; \alpha \in \mathbb{Z}_+\end{cases}$.

The Cauchy problem for the case of the Caputo fractional derivative presented in Equation (2) has the following structure:

$$\begin{cases}{}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{a^+,t}[x] & = f\left(t, x, {}_{\mathrm{c}}\mathfrak{D}^{\alpha_1}_{a^+,t}[x], {}_{\mathrm{c}}\mathfrak{D}^{\alpha_2}_{a^+,t}[x], \dots, {}_{\mathrm{c}}\mathfrak{D}^{\alpha_{m-1}}_{a^+,t}[x]\right) \\ x^{(k)}(a) & = b_k, \quad b_k \in \mathbb{C}, \quad k = 1,2,3,\dots,m.\end{cases} \quad (6)$$

where $0 < \mathbb{R}e(\alpha_1) < Re(\alpha_2) < Re(\alpha_3) < \cdots < Re(\alpha_{m-1}) < Re(\alpha)$ for $m \in \mathbb{Z}_+, m \geq 2$.

For this case, the initial conditions are given in terms of ordinary derivatives, which facilitates the physical interpretation.

Just as in the solutions of ordinary differential equations of integer order, the exponential function plays a fundamental role, an analogous function called the Mittag-Leffler function is also used in the solutions of differential equations of fractional order.

Definition 6: The Mittag-Leffler function of a parameter α is given by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

where $z \in \mathbb{C}$, $\mathbb{R}e(\alpha) > 0$. The Mittag-Leffler function of two parameters α and β is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

where $z, \beta \in \mathbb{C}$, $\mathbb{R}e(\alpha) > 0$.

When $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the one-parameter Mittag-Leffler function, that is, $E_{\alpha,1}(z) = E_{\alpha}(z)$. Moreover, when $\alpha = \beta = 1$, the function $E_{1,1}(z)$ is equivalent to the function e^z.

Proposition 1: If $0 < \alpha < 2$ and $\beta \in \mathbb{C}$ are arbitrary, for a $n \in \mathbb{Z}$ with $n \geq 1$, we have

$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} e^{z^{\frac{1}{\alpha}}} - \sum_{k=1}^{n} \frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k} + O\left(\frac{1}{|z|^{n+1}}\right)$ with $|z| \to \infty$, $|\arg(z)| \leq \alpha\frac{\pi}{2}$ and $E_{\alpha,\beta}(z) = -\sum_{k=1}^{n} \frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k} + O\left(\frac{1}{|z|^{n+1}}\right)$ with $|z| \to \infty$, $|\arg(z)| > \alpha\frac{\pi}{2}$.

The following results can be considered generalizations of the exponential function.

Definition 7: The function $e^{\lambda z}_{\alpha} = z^{\alpha-1}E_{\alpha,\alpha}(\lambda z^{\alpha})$ is called α −exponential function, where $z \in \mathbb{C}\backslash\{0\}, \lambda \in \mathbb{C}$ and $\mathbb{R}e(\alpha) > 0$.

According to Proposition 1 and Definition 6, it follows that $E_{1,1}(z) = E_1(z)$, and indeed, $e^{\lambda z}_1 = e^{\lambda z}$.

From Definition 7 of $\alpha-$exponential function and Proposition 1, we have the following result:

Proposition 2: If $0 < \alpha < 2$ and $z \in \mathbb{C}$, one has the following asymptotic equivalences for the $\alpha-$exponential function:

1. For $|\arg(z)| \leq \alpha\frac{\pi}{2}$, $e_{\alpha}^{\lambda z} \sim \frac{\lambda^{\frac{1-\alpha}{\alpha}}}{\alpha} e^{\lambda^{\frac{1}{\alpha}} z}$ when $|z| \to \infty$;
2. For $|\arg(z)| > \alpha\frac{\pi}{2}$, $e_{\alpha}^{\lambda z} \sim -\frac{1}{\lambda^2 \Gamma(-\alpha)} \frac{1}{z^{\alpha+1}}$ when $|z| \to \infty$.

3. Stability of Fractional Order Differential Equations

In this section, we consider the main results that have been presented on the stability of systems of ordinary differential equations of fractional order of linear, nonlinear, and time-delayed type.

Let us consider the linear system of fractional differential equations,

$$\mathbb{D}_{t_0,t}^{\tilde{\alpha}}[x(t)] = Ax(t), \tag{7}$$

where $x(t) = [x_1(t), x_2(t) \dots, x_n(t)]^T \in \mathbb{R}^n$, A is a matrix such that $A \in \mathbb{R}^{n \times n}$, $\tilde{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_n]^T$, $\mathbb{D}_{t_0,t}^{\tilde{\alpha}}[x] = \left[\mathbb{D}_{t_0,t}^{\alpha_1}[x_1], \mathbb{D}_{t_0,t}^{\alpha_2}[x_2], \dots, \mathbb{D}_{t_0,t}^{\alpha_n}[x_n]\right]^T$.

The derivative operator $\mathbb{D}_{t_0,t}^{\alpha_i}$ represents the derivative in the Caputo or Riemann-Liouville sense of order α_i, where $0 < i \leq 2$ for $i = 1,2, \dots, n$.

If $\alpha_1, \alpha_2, \dots, \alpha_n = \alpha$, Equation (7) can be written as

$$\mathbb{D}_{t_0,t}^{\alpha}[x(t)] = Ax(t) \tag{8}$$

The first results on stability for systems of linear differential equations were presented by Denis Matignon using tools from an algebraic approach and using asymptotic results applied to control theory [8]. This stability result was established for System (8) for order $0 < \alpha \leq 1$, which is presented below.

Theorem 1: In the autonomous system (8) with the Caputo derivative and initial value $x_0 = x(0)$, $0 < \alpha \leq 1$ is:

1. Asymptotically stable if and only if $|\arg[spec(A)]| > \alpha\frac{\pi}{2}$. For this case, the state components decay towards 0 as $\frac{1}{t^{\alpha}}$.
2. Stable if and only if either it is asymptotically stable or those critical eigenvalues, which satisfy $|\arg[spec(A)]| = \alpha\frac{\pi}{2}$, have geometric multiplicity, where $spec(A)$ denotes the eigenvalues of the matrix A corresponding to System (8).

Now, let us consider the fractional order ordinary differential equation presented in Equation (3); we can express it as

$$\mathbb{D}_{t_0,t}^{\tilde{\alpha}}[x(t)] = f\big(t, x(t)\big), \tag{9}$$

where $f: [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\tilde{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_n]^T$ with $m - 1 < \alpha_i < m$ for $m \in \mathbb{Z}_+$, and $i = 1,2, \dots, n$, $x_k = [x_{k_1}, x_{k_2}, \dots, x_{k_n}]^T \in \mathbb{R}^n$ for $k = 0,1,2, \dots, m-1$ represent suitable initial conditions, where $x(t) = [x_1(t), x_2(t) \dots, x_n(t)]^T \in \mathbb{R}^n$. The derivative operator denoted by $\mathbb{D}_{t_0,t}^{\tilde{\alpha}}$ represents the derivative in the Caputo or Riemann-Liouville sense.

If $\alpha_1, \alpha_2, \dots, \alpha_n = \alpha$, Equation (9) can be written as

$$\mathbb{D}_{t_0,t}^{\alpha}[x(t)] = f\big(t, x(t)\big) \tag{10}$$

and is called a system of fractional differential equations of the same order.

Definition 8: A constant vector x_* is said to be an equilibrium point of the system of fractional differential equation (10) if and only if $f(t, x_*) = \mathbb{D}_{t_0,t}^{\tilde{\alpha}}[x(t)]|_{x(t)=x_*}$ for all $t > t_0$.

Without loss of generality, we can consider the equilibrium point at the origin, that is, $x_* = 0$, and establish the following definition.

Definition 9: The solution $x_* = 0$ of the system of fractional order differential equation (10) is said to be:

1. Stable if for any initial condition $x_k = [x_{k_1}, x_{k_2}, \dots, x_{k_n}]^T \in \mathbb{R}^n$ with $k = 0,1,2, \dots, m-1$, there exists $\epsilon > 0$ such that any solution $x(t)$ of Equation (15) satisfies $||x(t)|| < \epsilon$ for all $t > t_0$.
2. Asymptotically stable if it is stable, and it is satisfied that $||x(t)|| \to 0$ when $t \to +\infty$.

[10, 11] studied the stability of systems of differential equations of nonlinear type using the Lyapunov functions for various generalizations of fractional derivative operators.

[10, 11] extended the exponential stability known in ordinary differential equations of integer order as the Mittag-Leffler stability and thereby introduced the direct Lyapunov method of fractional order using the fractional derivative operator in the sense of Riemann-Liouville and Caputo. To this end, the authors initially presented the following definitions [10-12].

Definition 10: A solution $x(t)$ of System (10) of nonlinear fractional differential equations is said to be Mittag-Leffler stable if it satisfies

$$\|x(t)\| \leq \{m[x(t_0)]E_{\alpha}(-\lambda(t - t_0)^{\alpha})\}^b \tag{11}$$

where E_{α} is the one-parameter Mittag-Leffler function, t_0 is the initial time, $\alpha \in (0,1)$, $\lambda \geq 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ are locally Lipschitzian on $x \in B \in \mathbb{R}^n$ with the Lipschitz constant m_0.

The same authors generalized Definition 10 as follows.

Definition 11: A solution $x(t)$ of System (10) of nonlinear fractional differential equations is said to be generalized Mittag-Leffler stable if it satisfies

$$\|x(t)\| \leq \left\{m[x(t_0)](t - t_0)^{-\gamma} E_{\alpha,1-\gamma}(-\lambda(t - t_0)^{\alpha})\right\}^b \tag{12}$$

where $E_{\alpha,1-\gamma}$ is the two-parameter Mittag-Leffler function, t_0 is the initial time, $-\alpha < \gamma \leq 1 - \alpha$, $\alpha \in (0,1)$, $\lambda \geq 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ are locally Lipschitzian on $x \in B \subset \mathbb{R}^n$ with the Lipschitz constant m_0.

The following results follow from the above definitions.

Remark 1: It follows from the properties of the Mittag–Leffler and completely monotonic functions that

$$\begin{aligned}\mathcal{D}^{\gamma}_{0,t}[E_\alpha(-\lambda t^\alpha)] &= \mathbb{D}^{\gamma}_{0,t}[E_\alpha(-\lambda t^\alpha)] \\ &= {}_{\mathrm{c}}\mathfrak{D}^{\gamma}_{a^+,t}[E_\alpha(-\lambda t^\alpha)] \\ &= t^{-\gamma}E_{\alpha,1-\gamma}(-\lambda t^\alpha)\end{aligned}$$

is a completely monotonic function for $\alpha \in (0,1)$, $\lambda \geq 0$, and $\gamma \in [0,1-\alpha]$.

Remark 2: The Mittag–Leffler stability and generalized Mittag–Leffler stability imply asymptotic stability.

Remark 3: Let $\lambda = 0$; it follows from Equation (40) that $\|x(t)\| \leq \left[\frac{m(x(t_0))}{\Gamma(1-\gamma)}\right]^b (t-t_0)^{-\gamma b}$, which implies that the power-law stability is a special case of the Mittag–Leffler stability.

Next, the extension of the direct Lyapunov method to the case of fractional order systems is presented, leading to the Mittag-Leffler stability [11].

Theorem 2: Let $x = 0$ be an equilibrium point of System (10) and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $L(t,x(t)): [0,\infty) \times D \longrightarrow \mathbb{R}$ be a continuously differentiable and locally Lipschitzian function with respect to x such that $\beta_1\|x\|^a \leq L(t,x(t)) \leq \beta_2\|x\|^{ab}$ and ${}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{0,t}L(t,x(t)) \leq -\beta_3\|x\|^{ab}$, where $t \geq 0$, $x \in \mathbb{D}$, $\alpha \in (0,1)$, and $a, b, \beta_1, \beta_2, \beta_3$ are arbitrary positive constants.

If the function $L(t,x(t))$ satisfies the previous inequalities, then it is said to be a fractional Lyapunov function and in effect the equilibrium point $x = 0$ is said to be Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, $x_* = 0$ is globally Mittag-Leffler stable.

Considering Remark 2, [10, 11] obtained an asymptotic stability result directly by weakening conditions in Theorem 2.

Theorem 3: Let $x = 0$ be an equilibrium point of System (10) and $D \subset \mathbb{R}^n$ be a domain containing the origin. Let $L(t,x(t)): [0,\infty) \times D \longrightarrow \mathbb{R}$ be a continuously differentiable and locally Lipschitzian function with respect to x such that $\beta_1\|x\|^a \leq L(t,x(t)) \leq \beta_2{}_{\mathrm{c}}\mathfrak{D}^{-\eta}_{0,t}\|x\|^{ab}$ and ${}_{\mathrm{c}}\mathfrak{D}^{\alpha}_{0,t}L(t,x(t)) \leq -\beta_3\|x\|^{ab}$, where $t \geq 0$, $x \in \mathbb{D}$, $\alpha \in (0,1)$, $\eta > 0, \eta \neq \alpha, |\alpha - \eta| < 1$, and $a, b, \beta_1, \beta_2, \beta_3$ are positive arbitrary constants. If the function $L(t,x(t))$ satisfies the previous inequalities, then it is said to be a fractional generalized Lyapunov function and in effect the equilibrium point $x_* = 0$ is asymptotically stable.

The asymptotic stability of System (10) can also be obtained through a particular type of functions, as presented below.

Definition 12: A continuous function $\alpha: [0,a) \longrightarrow [0,\infty)$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

Theorem 4: Let $x = 0$ be an equilibrium point of System (10). If it exists, a Lyapunov function $L(t,x(t))$ and class functions $\mathcal{K}$, α_1, α_2, and α_3 satisfy $\alpha_1\|x\| \leq L(t,x(t)) \leq \alpha_2\|x\|$ and ${}_{\mathrm{c}}\mathfrak{D}^{\beta}_{0,t}L(t,x(t)) \leq -\alpha_3\|x\|$, where $\beta \in (0,1)$. Then, the equilibrium point of System (10) is asymptotically stable.

The respective proofs to the above theorems can be consulted in [10, 11].

3.1. Stability of Systems of Fractional Order Differential Equations with Time Delay

The systems of fractional differential equations with time delay have been studied nowadays due to their applicability. Several models from different areas of knowledge, where the current development depends to a great extent on the history that has elapsed, are faithful reflections of systems of fractional differential equations with time delay.

In [23], the following system of fractional differential equations of the same order with multiple time delays was considered:

$$\begin{cases}\mathbb{D}^{\alpha}_{t_0,t}[x(t)] = A_0x(t) + \sum\limits_{i=1}^{m} A_ix(t-\tau_i) + B_0u(t), \\ 0 \leq \tau_1 < \tau_2 < \tau_3 < \cdots \tau_i < \cdots \tau_m = \Delta,\end{cases} \quad (13)$$

with the initial condition $x(t_0+t) = \psi(t) \in C[-\Delta,0]$, where $0 < \alpha < 1$, $\mathbb{D}^{\alpha_i}_{t_0,t}$ denotes either $\mathcal{D}^{\alpha_i}_{t_0,t}$ or ${}_{\mathrm{c}}\mathfrak{D}^{\alpha_i}_{t_0,t}$, $x(t) \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^l$ is a control vector, A_i for $i = 0,1,\ldots,m$, B_0 is a constant system matrix of appropriate dimensions, and $\tau_i > 0$ for $i = 1,2,\ldots,m$ is a pure time delay.

For the system of differential equations of fractional order (13) with time delay, we have the following definition.

Definition 13: The same order system (13) for $u(t) \equiv 0$ for all t, satisfying the initial condition $x(t_0+t) = \psi(t), -\Delta \leq t \leq 0$, is finite-time stable w.r.t. $\{t_0, J, \delta, \varepsilon, \Delta\}$ for $\delta < \varepsilon$ if and only if $\| \psi \|_c < \delta$, $\forall t \in J_\Delta, J_\delta = [-\Delta, 0] \in \mathbb{R}$ implies $\| x(t) \| < \varepsilon$, $\forall t \in J$, where δ is a positive real number, and $\varepsilon \in \mathbb{R}^+$, $\delta < \varepsilon$, $\| \psi \|_c = \max_{-\Delta \leq t \leq 0} \| \psi \|$, time interval $J = [t_0, t_0+T] \subset \mathbb{R}$, quantity T may be either a positive real number or a symbol $+\infty$.

Definition 14: The system given by (13) satisfying the initial condition $x(t_0+t) = \psi(t), -\Delta \leq t \leq 0$ is finite-time stable w.r.t. $\{t_0, J, \delta, \varepsilon, \Delta, \alpha_u\}$ for $\delta < \varepsilon$ if and only if $\| \psi \|_c < \delta$, $\forall t \in J_\Delta$, $J_\delta = [-\Delta, 0] \in \mathbb{R}$ and $\| u(t) \| < \alpha_u$, $\forall t \in J$ imply $\| x(t) \| < \varepsilon$, $\forall t \in J$, where δ is a positive real number, and $\varepsilon \in \mathbb{R}^+$, $\delta < \varepsilon$, $\alpha_u > 0$, $\| \psi \|_c = \max_{-\Delta \leq t \leq 0} \| \psi \|$, time interval $J = [t_0, t_0+T] \subset \mathbb{R}$, and quantity T may be either a positive real number or a symbol $+\infty$.

According to the above, we introduce sufficient conditions for finite-time stability.

Theorem 5:

1. A single-delay autonomous system given by Equation (13) for $u(t) \equiv 0$, $\forall t$, satisfying the initial

condition $x(t_0+t)=\psi(t), -\tau_1 \le t \le 0$, is finite-time stable w.r.t. $\{\delta,\varepsilon,\tau_1,t_0,J\}$, $\delta<\varepsilon$ if the following condition is satisfied:

$$\left[1+\frac{\sigma_{\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\right]\exp\left[\frac{\sigma_{\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\right]\le\frac{\varepsilon}{\delta},\quad \forall t \in j,$$

where $\sigma_{\max}(\cdot)$ is the largest singular value of matrix $(\cdot)$, namely $\sigma_{\max}^{A}=\sigma_{\max}(A_0)+\sigma_{\max}(A_1)$ and $0<\alpha<1$.

2. A multiple-delay autonomous system given by Equation (13) for $u(t)\equiv 0$ for all t, satisfying the initial condition $x(t_0+t)=\psi(t)$ for $-\Delta\le t\le 0$, is finite-time stable w.r.t. $\{\delta,\varepsilon,\Delta,t_0,J\}$, $\delta<\varepsilon$ if the following condition is satisfied:

$$\left[1+\frac{\sigma_{\sum\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\right]\exp\left[\frac{\sigma_{\sum\max}^{A}(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\right]\le\frac{\varepsilon}{\delta},\quad \forall t \in j,$$

where $\sigma_{\sum\max}^{A}(\cdot)$ is the largest singular value of matrix A_i for $i=0,1,2,\dots,n$.

The above stability results for linear fractional differential systems with time delay were obtained by applying the Bellman-Gronwall theorem [23, 24]. Then, [25] studied the finite-time stability of the single time delay nonautonomous system (13), presenting the following result.

Theorem 6: The linear single-delay nonautonomous system given by Equation (13) satisfying the initial condition $x(t_0+t)=\psi(t)$ for $-\tau\le t\le 0$ is finite-time stable w.r.t. $\{\delta,\varepsilon,\alpha_u,J_0\}$, $\delta<\varepsilon$ if the following condition is satisfied:

$$\left(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\right)E_{\alpha}(\sigma_{\max 01}t^{\alpha})+\frac{\nu_{u_0}t^{\alpha}}{\Gamma(\alpha+1)}\le\frac{\varepsilon}{\delta},\quad \forall t\in J_0=[0,T],$$

with $\| B_0\|\le b_0$, $\| u(t)\|\le\alpha_u$, $\nu_{u_0}=\alpha_u\frac{b_0}{\delta}$, and $\sigma_{\max}(\cdot)$ being the largest singular values of the matrix $(\cdot)$, where $\sigma_{\max 01}=\sigma_{\max}(A_0)+\sigma_{\max}(A_1)$.

Now, for the nonautonomous case, [26] also considered the following initial value problem:

$$\begin{cases}\mathcal{D}_{0,t}^{\alpha}[x(t)] &= A_0x(t)+A_1x(t_\tau)+f(t),\quad t\ge 0,\\ x(t) &= \phi(t),\ \ t\in[-\tau,0],\end{cases}\tag{14}$$

where $0<\alpha<1$, ϕ is a given continuous function on $[-\tau,0]$, A_0 and A_1 are constant system matrices of appropriate dimensions, and τ is a constant with $\tau>0$.

The system is defined over a time interval $J=[0,T]$, where T is a positive number, and $f(t)$ is a given continuous function on $[0,T]$.

Similarly, sufficient conditions for finite-time stability were derived by applying Gronwall's inequality.

Theorem 7: System given by (14) satisfying the initial condition is finite-time stable w.r.t $\{0,J,\delta,\varepsilon,\tau\}$, $\delta<\varepsilon$ if the following condition is satisfied:

$$\frac{(M+\mu_1)t^{\alpha}}{\Gamma(\alpha+1)}\exp\left[\frac{\mu_{\max}^{A}t^{\alpha}}{\Gamma(\alpha+1)}\right]\le\frac{\varepsilon}{\delta},\quad \forall t\in J,$$

where $M\ge\frac{\|f\|}{\|\phi\|}$, $\|\phi\|=\sup_{-\tau\theta\le 0}\|\phi(\theta)\|$, and $\mu_{\max}^{A}=\mu_{\max}(A_0)+\mu_{\max}(A_1)$ for $\mu_1=\mu_{\max}(A_1)$.

[27] first considered the analytical stability limit by using the special Lambert function W for a class of fractional differential equations with constant time delay as follows:

$$\mathcal{D}_{t_0,t}^{\alpha}[x(t)]=K_p x(t-\tau),\tag{15}$$

where α and K_p are real numbers, delay τ is a positive constant, and all the initial values are assumed to be zeros.

The authors derived a stability condition $\frac{r}{\tau}W\left(\frac{\tau}{r}(K_p)^{1/r}\right)\le 0$ by applying the Laplace transformation and the Lambert function $W(\cdot)$. Furthermore, they extended the analytical result to a class of delayed infinite dimensional systems described by the characteristic equation given by $(s+\alpha)^r-K_pe^{-\tau s}=0$; then, the stability condition is $\frac{r}{\tau}W\left(\frac{\tau}{r}e^{\frac{\tau\alpha}{r}}(K_p)^{1/r}\right)-\alpha\le 0$.

It is noteworthy that in [28], the results presented in [27] were improved, where the authors established the correct use of the Lambert W function for the stability analysis of a class of systems of fractional differential equations with time delay.

In [29], the stability of the following $n-$dimensional linear fractional differential equation system with multiple time delays is analyzed:

$$\begin{cases}{}_{c}\mathfrak{D}_{0,t}^{\alpha_1}[x_1(t)]=a_{11}x_1(t-\tau_{11})+\cdots+a_{1n}x_n(t-\tau_{1n})\\ {}_{c}\mathfrak{D}_{0,t}^{\alpha_2}[x_2(t)]=a_{21}x_2(t-\tau_{21})+\cdots+a_{2n}x_n(t-\tau_{2n})\\ \vdots\\ {}_{c}\mathfrak{D}_{0,t}^{\alpha_n}[x_n(t)]=a_{n1}x_n(t-\tau_{n1})+\cdots+a_{nn}x_n(t-\tau_{nn})\end{cases}\tag{16}$$

where $\alpha_i\in(0,1)$ for all $i=1,2,\dots,n$, the initial values $x_i(t)=\phi_i(t)$ are given for $-\max_{i,j}[\tau_{i,j}]=-\tau_{\max}\le t\le 0$ and $i=1,2,\dots,n$ defining time delay matrix $T=(\tau_{ij})_{n\times n}\in(\mathbb{R}_+)^{n\times n}$, coefficient matrix $A=(a_{ij})_{n\times n}$, state variables $x_i(t),x_i(t-\tau_{ij})\in R$, and initial values $\phi_i(t)\in C^0[-\tau_{\max},0]$.

Applying the Laplace transform and the final-value theorem, it follows the theorem below.

Theorem 8: If all the roots of the characteristic equation $\det(\Delta(s))=0$ have negative real parts, then the zero solution of System (16) is Lyapunov globally asymptotically stable, where

$$\Delta(s)=\begin{pmatrix}s^{\alpha_1}-a_{11}e^{-s\tau_{11}} & -a_{12}e^{-s\tau_{12}} & \cdots & -a_{1n}e^{-s\tau_{1n}}\\ -a_{21}e^{-s\tau_{21}} & s^{\alpha_2}-a_{22}e^{-s\tau_{22}} & \cdots & -a_{2n}e^{-s\tau_{2n}}\\ \vdots & \vdots & \ddots & \vdots\\ -a_{n1}e^{-s\tau_{n1}} & -a_{n2}e^{-s\tau_{n2}} & \cdots & s^{\alpha_n}-a_{nn}e^{-s\tau_{nn}}\end{pmatrix}.$$

The above result is applied to System (16) with $0<\alpha<1$, and the stability condition of (16) is:

1. If $K_p<0$, $(-K_p)^{\frac{1}{\alpha}}\ne-\frac{1}{\tau}\left[(2k+1)\pi-\frac{\alpha}{2}\pi\right]$;
2. $(-K_p)^{\frac{1}{\alpha}}\ne-\frac{1}{\tau}\left[(2k+1)\pi-\frac{\alpha}{2}\pi\right]$, where $k\in\mathbb{Z}$.

Then, the zero solution of System (16) is Lyapunov globally asymptotically stable.

[30] present a mechanism to construct a positive definite Lyapunov function L for a system of fractional differential equations without delay and then extend it to a system of fractional differential equations with delay. The authors present their results for the system:

$$ {}_{c}\mathfrak{D}_{t_0,t}^{\alpha}[x(t)] = f\big(x(t), x(t-\tau)\big) \qquad (17)$$

where $x(t) = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$ is the state vector. $f(x(t), x(t-\tau)) = [f_1(x(t), x(t-\tau_1)), f_2(x(t), x(t-\tau_2)), \dots, f_n(x(t), x(t-\tau_n))]$ is a nonlinear vector function satisfying the Lipschitz condition and the delay time $\tau = [\tau_1, \tau_2, \dots, \tau_n]^T \in \mathbb{R}^n$.

Theorem 9: With fractional order $0 < \alpha \le 1$, if there is a positive definite matrix P and a semi-positive definite matrix Q, for any $x(t) \in \mathbb{R}^n$, fractional system (17) satisfies

$$x^T(t) P_{\alpha}^{C}\, {}_{c}\mathfrak{D}_{t_0,t}^{\alpha}[x(t)] + x^T(t) Q x(t) - x^T(t-\tau) Q x(t-\tau) \le 0 \qquad (18)$$

The fractional delayed system (17) is Lyapunov stable.

For further details of the proof, see [30]. Also, in [31, 32], the construction of Lyapunov-type functions to establish the stability of systems of fractional differential equations with time delay is presented.

In [33], a direct extension of the stability criterion presented by Matignon for systems of fractional differential equations is made with respect to the inclusion of a delay. In the linear case, the fractional delay system

$$\begin{cases} {}_{c}\mathfrak{D}_{0,t}^{\alpha}[x(t)] = Ax(t-\tau), & t \in (0,\infty) \\ \quad x(t) = \phi(t), & t \in (-\tau, 0) \end{cases} \qquad (19)$$

where $0 < \alpha < 1$, $A \in \mathbb{R}^{n\times n}$ is a constant real $n \times n$ matrix, $x: (-\tau, \infty) \to \mathbb{R}^n$, $\phi(t) \in L^1([-\tau, 0])$, i.e., all components of $\phi(t)$ are absolutely Riemann integrable on $[-\tau, 0]$, and $\tau > 0$ is a constant real lag, may serve as the basic prototype of fractional differential equations.

In the scalar case, the asymptotic stability properties are

$$ {}_{c}\mathfrak{D}_{0,t}^{\alpha}[x(t)] = \lambda x(t-\tau), \quad t \in (0,\infty), \qquad (20)$$

where λ is a real number for using a transcendent inequality involving the fractional Lambert function.

Theorem 10: Let $\lambda \in \mathbb{R}$ for $0 < \alpha < 1$ and $\tau > 0$, then:

1. The zero solution of (20) is asymptotically stable if and only if $-\left(\frac{(2-\alpha)\frac{\pi}{2}}{\tau}\right)^{\alpha} < \lambda < 0$. Moreover, $x(t) \sim t^{-\alpha}$ as $t \to \infty$ for any solution $x(t)$ of (20).
2. The zero solution of (20) is stable if and only if $-\left(\frac{(2-\alpha)\frac{\pi}{2}}{\tau}\right)^{\alpha} \le \lambda \le 0$.

Also, in [33], Theorem 10 is the generalized introduction initially following the stable set

$$\mathcal{S}_{\alpha,\tau} = \left\{\lambda \in \mathbb{C}: |\lambda| < \left(\frac{|\arg(\lambda)| - \alpha\frac{\pi}{2}}{\tau}\right)^{\alpha}, |\arg(\lambda)| > \alpha\frac{\pi}{2}\right\},$$

where we assume $-\pi < arg(\cdot) \le \pi$ for its closure $cl(\mathcal{S}_{\alpha,\tau})$ and its boundary $\partial\mathcal{S}_{\alpha,\tau}$. Then, we have the following generalization of Theorem 10.

Theorem 11: Let $A \in \mathbb{R}^{n\times n}$, $0 < \alpha < 1$, and $\tau > 0$, then:

1. The zero solution of (20) is asymptotically stable if and only if all eigenvalues λ of A are located inside $\mathcal{S}_{\alpha,\tau}$. Moreover, $\| x(t) \| \sim t^{-\alpha}$ as $t \to \infty$ for any solution $x(t)$ of (20).
2. The zero solution of (20) is stable if and only if all eigenvalues λ of A belong to $cl(\mathcal{S}_{\alpha,\tau})$ and all eigenvalues lying on $\partial\mathcal{S}_{\alpha,\tau}$ have the same algebraic and geometric multiplicities.

For a more in-depth study of the subject, see [34-37] on studying the stability of systems of fractional differential equations with time delay, where the methods of stability in the sense of Lyapunov are extended to the case of fractional differential equations with time delay.

4. Conclusion

The study of the stability of systems of differential equations of fractional order is one of the fundamental topics in the study of various models of applied sciences and engineering.

In this paper, we present the fundamental contributions to the study of stability from the first results presented by Matignon to the present time. We attempt to present fundamental results of studying the stability of systems of linear, nonlinear, and fractional time-delay differential equations.

With this study, we note that significant progress has been made in the construction of a grounded stability theory for fractional differential equations.

Regarding the stability of systems of fractional differential equations with time delay, fundamental tools have been provided to generalize the stability concepts of linear and nonlinear differential equations, such as the use of fractional Lyapunov functions.

This work is novel because it aims to contribute to the current debate in the literature on the stability of systems of fractional order differential equations with time delay, which are of vital importance in the modeling of various problems in applied sciences. The non-local character of the fractional order operators with the time delay allows obtaining a better response in the modeling of the systems.

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