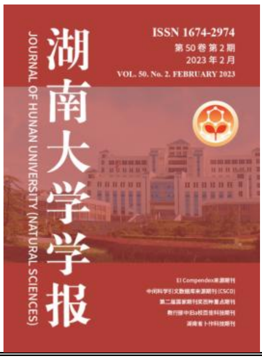

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The Generalization as a Tool to Develop the Algebraic Thinking Process

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Abstract: In this paper, we propose a reflection on how pupils use algebraic generalization and symbolization during the transition from a geometric thinking process to the algebraic one. This paper describes a new method/idea based on the use of geometric patterns in some proposed learning situations with the primary aim of understanding how this use may enable learners to develop and improve their algebraic thinking process. Thus, we proposed to a sample of pupils from 10 to12 years old to draw some geometric figures of squares and rectangles and asked them to conjecture the three well-known remarkable identities, $(a+b)^2$, $(a-b)^2$ and a^2-b^2. The students are supposed to put these squares and rectangles together to form a single square or a rectangle and to deduce these identities. Our analysis focuses on videos capturing student behavior using patterns to develop some generalization of the algebraic process on data collected during student interviews and on the difficulties of students deriving algebraic formulas from some geometric patterns.

Keywords: generalization, algebraic thinking process, geometric thinking process, patterns in education.

概括作為發展代數思維過程的工具

摘要：在本文中，我們對學生在從幾何思維過程到代數思維過程的過渡過程中如何使用代數概括和符號化提出了反思。本文描述了一種新的方法/想法，該方法/想法基於在某些建議的學習情境中使用幾何圖案，主要目的是了解這種使用如何使學習者能夠發展和改進他們的代數思維過程。因此，我們建議10至12歲的學生樣本繪製一些正方形和長方形的幾何圖形，並要求他們猜想三個眾所周知的顯著恆等式，(a加b)^2，(a減去b)^{2}和a^{2}減去b^2。學生應該將這些正方形和長方形放在一起形成一個正方形或長方形，並推導出這些恆等式。我們的分析側重於使用模式捕捉學生行為的視頻，以對學生訪談期間收集的數據進行代數過程的一些概括，以及學生從某些幾何模式中推導出代數公式的困難。

关键词：概括，代數思維過程，幾何思維過程，教育模式。

1. Introduction

From the perspective of developing an algebraic thinking process, the generalization phase in a didactic situation represents the heart of the construction of mathematical knowledge in algebra. [18] emphasizes that generalization is an essential process in any mathematical activity and, in particular, in algebra". [1] goes even further by highlighting that this activity is a characteristic constant of the algebraic thinking process. [5] considers the generalization as a characterization of the algebraic competence assessed through the ability to express general numerical relations. Indeed, the algebraic language as a tool for formulating problems and solving them plays an essential role in helping students develop their algebraic thinking process and deepen their mathematical rationality as they go along. According to [11], in the learner's algebraic register, the generalization activities are a more efficient tool to initiate the students to the algebraic culture. [12] considered them opportunities to develop the students' algebraic thought process since they favor the mathematical reasoning identified regularities and then express it with some algebraic formulas using mathematical symbols. [9] considers symbolism a tool to express generalities and points out that the algebraic denotations made it possible to pass from the particular to the general and thus raise the algebraic thought to a level far superior to that of the arithmetic one considered ordinary. Indeed, algebraic denotations can access geometric objects (perimeters, areas, volumes, etc.) and manipulate them. It is this type of exchange that has allowed mathematics to develop and progress as a powerful instrument in scientific research.

In this context, we propose to analyze the process of the algebraic generalization of a sample of middle school students. We propose to them some activities based only on geometric patterns (squares and rectangles). The aim is to deduce the three remarkable identities $(a+b)^2$, $(a-b)^2$, and a^2-b^2. The analysis of the pupils' reactions (filmed during these activities) reveals the difficulties they encounter in an algebraic meaning to a geometric pattern that asks us to analyze the transition from a geometric thinking process to an algebraic one.

2. Theoretical Framework

Algebra is an essential element in the science curriculum of the students and often serves as a background for other science courses. To help students in middle school to overcome their difficulties in algebra, several researchers suggest developing early algebraic thinking since elementary school; we then talk about “Early Algebra” [8, 16]. “Early Algebra” does not necessarily mean algebra early, in other words, teaching algebra in primary rather than secondary school, nor preparation for it. It is a strategy for enriching the mathematical content taught in elementary school by offering students opportunities to develop their algebraic thought and further deepen certain mathematical notions and concepts (the concepts of operation, equality, equation, regularity, formula, variable, and variation, among others) [2].

It is only in secondary school that educational systems give real meaning to teaching some algebraic concepts. According to [10], they offer a promising opportunity to develop the algebraic thought process. In elementary school, through his various mathematical activities, the pupil was introduced, unwittingly, to prerequisites for algebra, in particular, the search for missing terms by using the properties of operations and the relations between them, the appropriation of the meaning of relations of equality and equivalence, the use of priorities of operations, and the search for regularity in different contexts. According to [15], the primary characteristic of algebraic thinking is the behavior of these operations. According to [13], algebraic thinking has the following three characteristics.

• The indeterminacy, which relates to the ability to exploit problems that involve unknown numbers;

• The denotation, consisting of succeeding in naming or symbolizing these unknown numbers. This denotation can be done in different ways, using the alphanumeric code, but also natural language, gestures, or unconventional signs;

• The analytical reasoning leads to treating some indeterminate quantities as if they knew and to succeed in operations on these unknown numbers.

Additionally, algebraic thinking covers some aspects related, on the one hand, to a particular form of thought (analytical reasoning) and, on the other hand, to symbolizing this thought (denotation). Regarding the algebraic generalization approach, [14] observes that the students can propose three types of written symbolization, corresponding to three types of algebraic generalizations: factual, contextual, and symbolic. In the case of actual algebraic generalization, the student symbolizes the unknown from a number: he chooses a number and applies the formula starting from it. In the contextual one, the unknown uses a symbolic substitute (question mark, empty frame, letter, etc.), but the elaborate symbolization retains traces of the situation that fostered it. For example, the student uses parentheses to indicate the order of the operations without this being necessary according to the rule of precedence of operations. The symbolic algebraic generalization stands out from the situation to offer perfectly mathematically correct writing, which is no longer rooted in the context of the sequel [21]. A generalization is said to be algebraic when the generality produced can be represented in an algebraic register, for example, by an expression involving operations, numbers, letters, words, and symbols. For [3, 19], generality comes from the recognition of regularity, and to generalize is to identify/recognize a regularity. These authors characterize the algebraic generalization according to the type of associated reasoning, meaning that the algebraic generalization identifies a regular pattern based on the observation of known terms. This approach is called “abduction.” Inference uses the observed regularity to produce an expression for any term in the sequence, whether near or far from a known term.

Because of its didactic interest, we are especially interested in the resolution of the so-called generalization problems, that is to say, problems whose solution leads to the implementation of generalization, the identification of a model, following the approach [23]. Indeed, our work is a part of this research perspective to study the transition from geometric thinking to algebraic one, which in particular causes students to find the algebraic aspect by using some generalization activities that ask them to navigate their imagination to develop their algebraic thinking through some experiments in the form of geometric situations formulated the remarkable algebraic identities $(a+b)^2$, $(a-b)^2$, or a^2-b^2. To this end, it seems to us that the main questions that can arise are the following:

• How can early secondary students foster algebraic thinking as a generalization emerges?

• What roles do reasoning and symbolization play?

• What validations can be implemented?

• Would algebraic generalization activities improve the progression of reasoning and symbolization?

• More broadly, what processes of generalization and symbolization do students develop during the pattern-based "remarkable personalities" activity?

We then propose an exploratory work relative to the previous questions consisting of analyzing with various didactic approaches a situation of generalization in a first class of middle school based on some geometric patterns that involve and lead to an algebraic thinking process.

3. Methods

We worked on three generalization activities on three well-known remarkable identities proposed through some geometric patterns (see Fig 1). The observation of the spontaneous reactions of the students, who were filmed during all stages of these activities, enabled us to analyze the potential of this series of three activities in the development of generalizing algebraic thinking by students in the first year of middle school (from 11 to 13 years old).

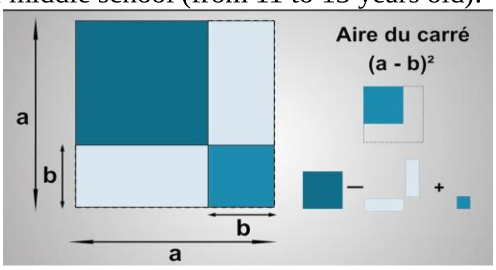

Fig. 1 The first remarkable identified pattern

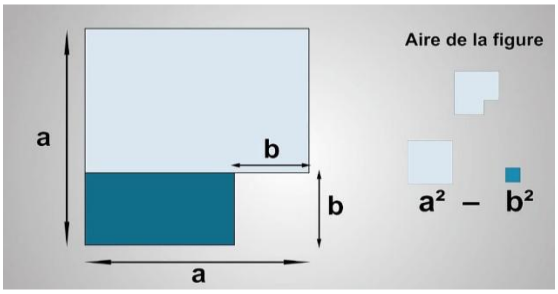

Fig. 2 The second remarkable identified pattern

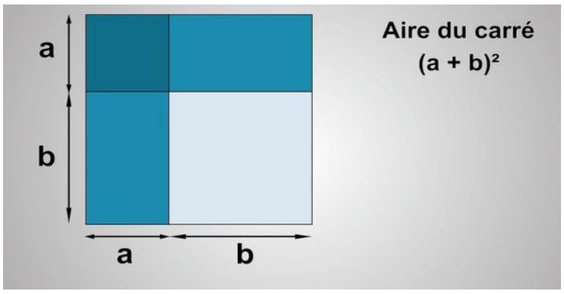

Fig. 3 The third remarkable identified pattern

The three problems are about the remarkable identities $(a+b)^2$; $(a-b)^2$; a^2-b^2 in a figurative context giving to the students the opportunity to respond with geometric patterns. So, all students are involved in solving problems to conjecture a purely algebraic equation. It is quite appropriate to get students to establish remarkable equality using their geometric register (equality of areas). This is what Al-Khawarizmi, an outstanding Persian mathematician, originally used to justify his algebraic transformations: the proof by figures. Thus, our experimentation based on a problem-situation teaching method does not begin with a lesson that explains the course but rather with problems that the students must solve to deduce knowledge. Thus the pupils are in a situation of producers of the knowledge and not reproducing the knowledge of their teacher. The experimentation involved several activities:

• In the first activity, we propose to the students some activities on the distributive property $(a+b)(c+d)$;

• In the second activity, we ask the students to formulate the remarkable identities $(a+b)^2$, $(a-b)^2$, and a^2-b^2;

• In the third activity, we draw the first image as in Fig. 1 and ask the students to deduce from it, by manipulating geometric areas, the first remarkable identity $(a+b)^2$;

• In the fourth activity, we ask a group of students to deduce the second remarkable identity $(a - b)^2$ from the first remarkable one $(a+b)^2$;

• In the fifth activity, we ask another group of students to inspire the third activity and then deduce the second remarkable identity $(a-b)^2$ by proposing suitable geometric patterns;

• In the sixth activity, we ask a group of pupils to deduce freely and without any indication the third remarkable identity a^2-b^2;

• In the seventh activity, we propose some application exercises.

The sessions were filmed (see Fig. 2) from a fixed location at the back of the classroom with a camera directed toward the blackboard. The sessions included the collection of draft sheets of the students (see Fig. 3). The teacher, except during the 3rd activity, had always remained in a withdrawn position, sometimes explaining the instructions while moving between the students to observe their "spontaneous" reactions. The problem statements were given without any particular indication. For the 3rd activity, and more clarity, the different zones took numbers while asking the students to express the area of each zone as a function of “a” and/or “b” and the area of the combination of zones. Then the students were asked to determine using the equality of their remarkable identity $(a+b)^2$ and write it in their notebook to know to what extent the students could understand the lesson by this method. Another activity that aimed at other remarkable identities in the geometrical approach had a promising effect on the passage of their geometrical thinking to an algebraic one, and as the purpose of these sessions is to have a generalization on the course based on the patterns that will allow them to structure their vision, and therefore to model their way of apprehending the space that surrounds us, to move from geometric language to the algebraic one through an activity of algebraic generalization. The latter is an exercise to the chain on the area of rectangles in different ways, which leads students to connect their ideas to have a chain that helps them find equal expressions and therefore have properties concerning remarkable identities ($a+b)^2$; $(a-b)^2$; a^2-b^2.

4. Results

After analyzing the answers of the students on the proposed activity and the exercises requested, the students first had difficulties dealing with this lesson in a geometric framework since they are used to having to justify the expression algebraically. Also, they had difficulties in gathering the expressions of the areas and analyzing the equality between the figures of the same size. Hence, the passage from a purely geometrical thinking process to deal with an algebraic problem and find the algebraic properties was a foreign sequence and a new way of explaining for them. This allows them to think, orient, grope, respond, and then answer questions that seek connections between the activity and the property they want to find.

Figs. 4-7 Some screenshots of the activities

For example, we found some students having difficulties understanding the question concerning the calculation of area in two ways by distributing the figure, which was a blockage at the level of processing of the figure. After the explanation of the method, the students started the realization of the suggested rule. As for other students who found it difficult to combine the two methods, "global and by distribution," to find equality of areas, the obstacle related to the difficulty of combining geometry and algebra. After the explanation, the students better assimilated the method to find the correct answer. As a result, the students reacted well to the explanations.

For example, consider the student who made the following error:

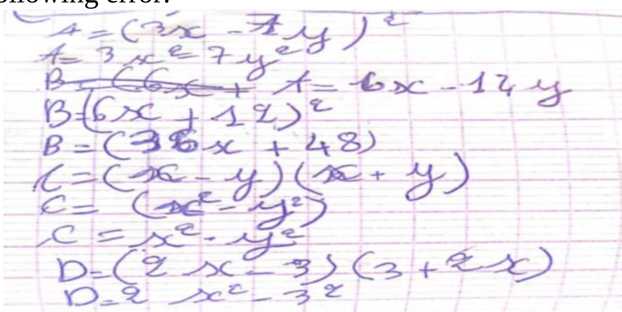

Fig. 8 Screenshot of the student production

He has made many mistakes, such as the false application of the property of the first and second remarkable identity in A and B. The wrong application of the power properties, for example, he wrote in A:

$(3x)^2 = 3x^2$

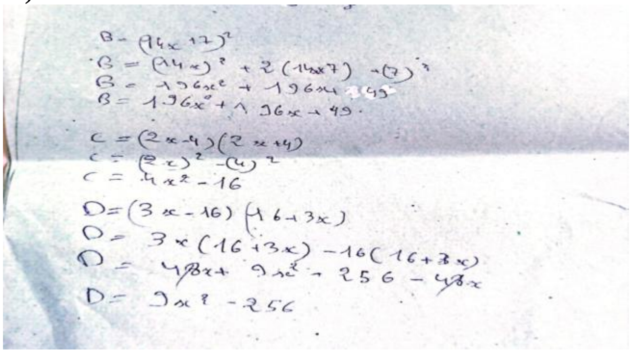

Fig. 9 Screenshot of the student production

For this student, we can detect some difficulties in applying the third remarkable identity in D or in the use of the classical distribution rules.

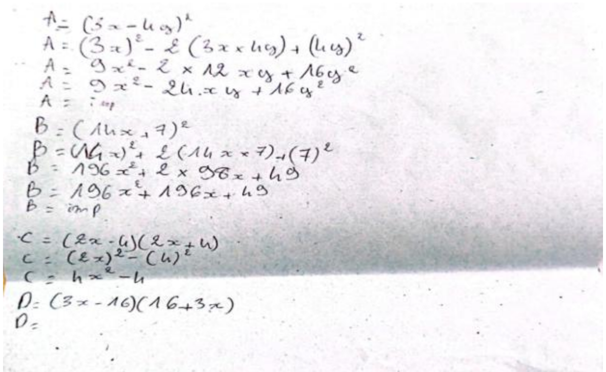

Fig. 10 Screenshot of the student production

This production reveals that the student wants to resolve an equation when he writes the term "impossible" in the last step of each operation. Also, the change in the places of the terms poses this student confusion in the application of the third remarkable identity.

5. Discussion

Let us briefly report the usual students' way of working during the transition from a geometrical thinking process to one. On the one hand, the students are confronted with the resolution of equality of areas, including the notion of the area that appears as tools to have expressions, to discover. On the other hand, exercises in the manipulation of specific algebraic expressions are offered. Both ways occur under the control of "transformation rules" that the student must remember. From our perspective, this type of geometric work with algebraic expressions is powerful for a beginner to construct these objects with meaning. Without forgetting the transition from geometrical thinking to algebraic thinking through the generalization of such a fate, it gives students a relevant tool that promotes the specific aspect of algebraic thinking.

Generalization activities based on schematic support provide a rich context for working on algebraic thinking with students in both primary and secondary schools. The main interest of this type of activity lies in the diversity of possible expressions of generalization. Indeed, the different possible visualizations of the figures will lead to different calculations and expressions according to the regularity identified. The different expressions of the generalization resulting from these visualizations will promote rich interactions [20]. For [22], communication and social interactions are considered consubstantial with learning and thinking. Cognitive development originates in mediated social interactions. For [12], the emergence, during teacher-student interactions, of formal mathematical knowledge, such as the algebraic method of solving equations, represents more than an exchange of points of view. This type of relationship has a deep epistemic value. It is indeed a complex and dialectical process in which the learning process is closely connected to that of teaching. For [12, 4], this involves teacher interventions such as reformulating, noting, interpreting, underlining, suggesting new elements, etc. These interventions make it possible to "speak in mathematical language" using various types of signs, such as representations, conventional symbols, informal notations, verbal language, drawings, gestures, and objects [17]. Additionally, students will be able to argue the ways of generalization found based on drawings.

During the sessions with the students, the subject of studying remarkable identities was, first of all, the one of individual work, which consisted of research to have more knowledge and developing activities. Following this, write the solution in everyone's notebook. Massively used during individual research for the problem study, to evaluate the progression of the treatment of the problem by the pupils from where the results of this treatment could consider some characteristics of written production that arise in the work of students regarding the different faults of the activities:

- Distribution fault: $(a+b)(c+b)=a.c+b.b$; the student here could not distribute all the terms;
- For the lack of the first formula "Fig. 1": the student has found the total area of the square $(a+b)^2$ and a difficulty in gathering the areas divided into parts; connect the ideas between the areas found in the figure of the formula $(a+b)^2$ to find the first remarkable
- Success rate between activities 2 and 3: after the passage of the proposals requested algebraically to the geometric explanation and the results obtained by the pupils with the correction of their errors in parallel, the pupils could have a very advanced vision of the remarkable identities to the purpose of imagining the formula in the geometric aspect; also, they quickly perceived and firmly anchored in their head.

Written productions that arise in the work of students regarding the proposed exercise are:

- Use the expressions retained previously to solve the problem;

- The appreciation of the transition from the geometric approach to the purely algebraic approach.

The entry using this kind of activity made it possible to give meaning to the expression of the formulas and the generality, considering the difficulties that appeared when analyzing the reasoning of some students in their interpretations and their answers. However, this does not prevent the passage from geometric thinking from giving learners the use of geometric patterns to activate the automatism of their algebraic thinking for generalizing algebraic formulas. Indeed, the tendency to generalize and the tendency to reason about the unknown favor the tendency to symbolize [18].

6. Conclusion

We can conclude that mathematics is a discipline that gives students the most difficulty since mathematical objects have a complicated and abstract nature and sometimes make their accessibility possible only through their various representations. Hence, the crucial role belongs to using and understanding mathematical symbols in learning.

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