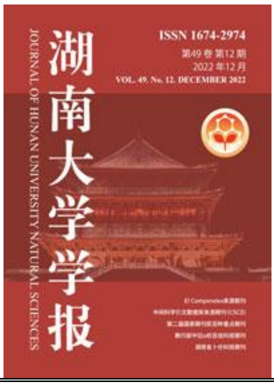

湖南大学学报（自然科学版）
Journal of Hunan University (Natural Sciences)

第 49 卷 第 12 期
2 0 2 2 年 12 月

Available online at http://jonuns.com/index.php/journal/index

Vol. 49 No. 12
December 2022

Open Access Article https://doi.org/10.55463/issn.1674-2974.49.12.27

On the Stability of the van der Pol-Mathieu-Duffing Oscillator under the Effect of Fast Harmonic Excitation

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Received: August 7, 2022 / Revised: October 4, 2022 / Accepted: November 10, 2022 / Published: December 30, 2022

Abstract: This paper aims to examine the nonlinear dynamics of a van der Pol-Mathieu-Duffing oscillator under the effect of fast harmonic excitation. The governing equations of motion describing the harmonically forced oscillations of the van der-Pol-Mathieu-Duffing oscillator are expressed in terms of the second-order nonhomogeneous nonlinear ordinary differential equation with suitable initial conditions. This paper uses Krylov-Bogoliubov averaging technique for the stability analysis of the system. The frequency response curves under the effect of external excitation, damping, and nonlinearity are obtained at various resonances. Additionally, the stable and unstable regions were identified. It turns out that the damping reduces the amplitude of oscillations and squeezes the instability regions, whereas the stability region grew with the increase in the amplitude of external excitation.

Keywords: Krylov-Bogoliubov averaging, resonances, frequency response curve, stability.

快諧波激勵作用下范德波爾-梅西-杜芬振盪器的穩定性

摘要：本文旨在研究范德波爾-梅西-杜芬振盪器在快諧波激勵作用下的非線性動力學。描述范德波爾-梅西-杜芬振盪器的諧波強迫振蕩的運動控制方程用具有合適初始條件的二階非齊次非線性常微分方程表示。本文采用克雷洛夫-博戈柳博夫平均技術對系統進行穩定性分析。在各種共振條件下得到了在外部激勵、阻尼和非線性作用下的頻率響應曲線。此外，還確定了穩定和不穩定區域。事實證明，阻尼減小了振蕩的幅度並擠壓了不穩定區域，而穩定區域隨著外部激勵幅度的增加而增加。

关键词：克雷洛夫-博戈柳博夫平均、共振、頻率響應曲線、穩定性。

1. Introduction

The van der Pol-Mathieu-Duffing Oscillator has a wide range of applications in diverse fields for instance, in mechanics, biology and epidemiology. Mathematically, the van der Pol-Mathieu-Duffing oscillator is modeled as nonlinear second order ordinary differential equations. The van der Pol-Mathieu-Duffing oscillator [1] is modeled as:

$$\ddot{x} + (1 - h\cos\omega t)x - (\alpha - \beta x^2)\dot{x} - \gamma x^3 = a\Omega^2 \cos x \cos \Omega t, \quad (1)$$

Subject to initial conditions:

$$x(0) = x_0, \ \dot{x}(0) = x_1. \quad (2)$$

The coefficients α, β are damping coefficients, γ is a nonlinearity term, h and a are excitation amplitude and taken to be small, x_0 and x_1 are initial displacement and velocity. The dots here denote the differentiation with respect to time t. The fast harmonic excitation frequency Ω is considered as large compared to ω such that resonance phenomena with the frequency Ω are avoided. The mentioned model is a mixture of three equations, that is, van der Pol, Mathieu and Duffing equation.

2. Literature Review

This system has applications in fast harmonic excitation on chaotic dynamic [2], mechanics, biology, and epidemiology [3], the dynamical behavior of dust grain charge in dusty plasmas [4], modeling micro-electro-mechanical system (MEMS) devices [4], and electrical engineering. Many real life necessities like an inverted pendulum, Duffing oscillator showing the novel spring, microelectro mechanical system devices, and optical parametric oscillations which are mathematically termed as the Van der Pol-Mathieu-Duffing oscillator. The dynamics of such an oscillator have been examined through quantitative and qualitative approaches. In the quantitative approach, the perturbation method [5–6], homotopy analysis method [7–8], Runge Kutta methods [9-10], the finite difference and finite element methods are used [11–12]. In the qualitative approach [13–15], the stability of the system is investigated in terms of periodic steady state, the first integral and bifurcation analysis. In [16–17] the authors studied the van der Pol-Mathieu-Duffing equation analytically and numerical methods under the influence of harmonic excitation. In [18] the authors studied the van der Pol-Mathieu-Duffing oscillator under the influence of fast harmonic excitation by the Melnikov method. It has been observed that the chaotic domain in the parameter space can be significantly reduced for some parameters of fast excitations.

3. Method

In this paper, nonlinear dynamics of the van der Pol-Mathieu-Duffing oscillator under the effect of fast harmonic excitation are studied using Krylov-Bogoliubov averaging method. The frequency response curves are obtained for different cases and for different resonances. The stability of the system at different resonance cases was examined in detail. Finally, the effect of different physical parameters on the system motion was analyzed graphically.

4. Results – Solution to the Problem

In this section, the frequency-response curve for the governing equations of motion given in Eq. (1) and (2) by the application of Krylov-Bogoliubov averaging method is constructed. Let us consider that the parameters h, α, β, γ and a given in Eq. (1) are to be of order ϵ $(0 < \epsilon \ll 1)$. The direct application of the averaging technique is not easy due to the inclusion of the term $\cos x$ in excitation. We therefore use the Maclaurin series for the expansion of $\cos x$. We write

$$\cos x \cong \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \quad (3)$$

4.1. Case I: $\cos x \cong 1$

By putting $\cos x \cong 1$ and by taking $h, \alpha, \beta, \gamma, a = O(\epsilon)$ into Eq. (1), we get

$$\ddot{x} + x = \epsilon\left[xh\cos\omega t + (\alpha - \beta x^2)\dot{x} + \gamma x^3 + a\Omega^2 \cos \Omega t\right] \quad (4)$$

For further simplification, we let $\omega t = s$ and $\omega^{-2} = 1 - \epsilon\mu$ so that

$$\frac{dx}{dt} = \omega\frac{dx}{ds} \text{ and } \frac{d^2x}{dt^2} = \omega^2\frac{d^2x}{ds^2} \quad (5)$$

By setting Eq. (5) into Eq. (4) it yields:

$$\frac{d^2x}{ds^2} + x = \epsilon\left[\mu x + xh\cos s + (\alpha - \beta x^2)\frac{dx}{ds} + \gamma x^3 + a\Omega^2 \cos \Omega s\right] \quad (6)$$

Let us consider the solution of Eq. (6) is assumed in the form as under:

$$x(s) = b(s)\cos(s + \psi(s)) \quad (7)$$

By keeping b and ψ constant, the differentiation of (7) with respect to s yields

$$\dot{x} = -b\sin(s + \psi(s)) \text{ and } \ddot{x} = -b\cos(s + \psi(s)) - b\dot{\psi}\cos(s + \psi(s)) - \dot{b}\sin(s + \psi(s)) \quad (8)$$

By plugging Eq. (7) and (8) into Eq. (6) and equating the resulting equations for $\dot{b}$ and $\dot{\psi}$, we obtain

$$[\dot{b} = -\epsilon\mu b\cos(s+\psi)\sin(s+\psi) + bh\cos(s)\cos(s+\psi)\sin(s+\psi) - b\alpha\sin^2(s+\psi) + b^3\beta\sin^2(s+\psi)\cos^2(s+\psi) + b^3\gamma\cos^3(s+\psi)\sin(s+\psi) + a\Omega^2\cos\Omega t\sin(s+\psi)] \tag{9}$$

$$\dot{\psi} = -\epsilon\,[\mu\cos^2(s+\psi) + h\cos(s)\cos^2(s+\psi) - \alpha\sin(s+\psi)\cos(s+\psi) + b^2\beta\sin(s+\psi)\cos^3(s+\psi) + b^2\gamma\cos^4(s+\psi) + \tfrac{a}{b}\Omega^2\cos\Omega\cos(s+\psi)] \tag{10}$$

It can be observed that Eq. (9) and (10) are 2π periodic in s. and the average of (9)-(10) yields.

$$\dot{b} = \frac{1}{2}\epsilon\, b\,\alpha - \frac{1}{8}\epsilon\, b^3\beta + \frac{\Omega^2}{2}\epsilon a\sin[(\Omega-1)-\psi] \tag{11}$$

$$\dot{\psi} = -\frac{1}{2}\epsilon\,\mu - \frac{3}{8}\epsilon b^2\gamma - \frac{\Omega^2}{2b}\epsilon a\cos[(\Omega-1)-\psi] \tag{12}$$

If we choose $\Omega \cong 1$, and put $\Omega - 1 = \epsilon\sigma$ in (11) and (12) it yields:

$$\dot{b} = \frac{1}{2}b\,\alpha - \frac{1}{8}b^3\beta - \frac{1}{2}a\sin\eta \tag{13}$$

$$\dot{\eta} = \sigma - \frac{1}{2}\mu - \frac{3}{8}b^2\gamma - \frac{1}{2b}a\cos\eta \tag{14}$$

where $\eta = \sigma s - \psi$.

For a steady state, we make $\dot{b}$ and $\dot{\eta}$ to zero into Eq. (13) and (14) respectively, thus we get

$$\frac{1}{2}b\,\alpha - \frac{1}{8}b^3\beta - \frac{1}{2}a\sin\eta = 0 \tag{15}$$

$$\sigma - \frac{1}{2}\mu - \frac{3}{8}b^2\gamma - \frac{1}{2b}a\cos\eta = 0 \tag{16}$$

Dividing Eq. (15) by "b", we get

$$\frac{1}{2}\alpha - \frac{1}{8}b^2\beta - \frac{1}{2b}a\sin\eta = 0 \tag{17}$$

Now squaring and adding equations (16) and (17) it yields:

$$\sigma = -\frac{1}{2}\mu - \frac{3}{8}b^2\gamma \pm \frac{1}{2}\left[\frac{a^2}{b^2} - \alpha^2 + \frac{1}{2}b^2\alpha\,\beta - \frac{1}{16}b^4\beta^2\right]^{\frac{1}{2}} \tag{18}$$

Let us put $\sigma = \frac{1}{\epsilon}(\Omega - 1)$ into Eq. (18):

$$\Omega = 1 + \epsilon\left[-\frac{1}{2}\mu - \frac{3}{8}b^2\gamma \pm \frac{1}{2}\left[\frac{a^2}{b^2} - \alpha^2 + \frac{1}{2}b^2\alpha\beta - \frac{1}{16}b^4\beta^2\right]^{\frac{1}{2}}\right] \tag{19}$$

Eq. (19) represents the frequency-response curve for this case $\cos x \cong 1$.

4.1.1. Stability Analysis of the System

To examine the stability of the system, we shall find the eigenvalues. To do so, we compute the Jacobean of the coupled system of ordinary differential equations (15)-(16) as under:

$$J = \begin{pmatrix} \frac{1}{2}\alpha - \frac{3}{8}b^2\beta & -\frac{1}{2}a\cos\eta \\ -\frac{3}{4}b\gamma + \frac{1}{2b^2}a\cos\eta & \frac{1}{2b}a\sin\eta \end{pmatrix} \tag{20}$$

The critical points p of the averaged Eqs. (13) and (14) satisfy the following transcendental equations:

$$\frac{b\alpha}{2} - \frac{1}{8}b^3 = a\sin\eta \tag{21}$$

$$\sigma - \frac{\mu}{2} - \frac{3}{8}b^2\gamma = \frac{a}{2b}\cos\eta \tag{22}$$

With Eq. (21) and (22), Eq. (20) becomes:

$$J = \begin{pmatrix} \frac{1}{2}\alpha - \frac{3}{8}b^2\beta & -b\sigma + \frac{b\mu}{2} + \frac{3b^3\gamma}{8} \\ \frac{\sigma}{b} - \frac{\mu}{2b} - \frac{9b\gamma}{8} & \frac{1}{2}\alpha - \frac{1}{8}b^2\beta \end{pmatrix}_p \tag{23}$$

For eigenvalues $|J - \lambda I| = 0$ where I is the identity matrix of order 2×2, thus the Eigenvalues can be obtained as under:

$$\begin{vmatrix} \frac{1}{2}\alpha - \frac{3}{8}b^2\beta - \lambda & -b\sigma + \frac{b\mu}{2} + \frac{3b^3\gamma}{8} \\ \frac{\sigma}{b} - \frac{\mu}{2b} - \frac{9b\gamma}{8} & \frac{1}{2}\alpha - \frac{1}{8}b^2\beta - \lambda \end{vmatrix} = \tag{24}$$

Simplification of Eq. (24) yields the following eigenvalues of the system:

$$\lambda_{1,2} = \left[\frac{1}{2}\alpha - \frac{1}{4}b^2\beta\right] \pm \left[\frac{b^4\beta^2}{16} - \left\{\left(\sigma - \left(\frac{\mu}{2} + \frac{3b^2\gamma}{8}\right)\right)\left(\sigma - \left(\frac{\mu}{2} + \frac{9b^2\gamma}{8}\right)\right)\right\}\right]^{\frac{1}{2}} \tag{25}$$

Note that instability occurs in the system when $\lambda > 0$ for a stable system $\lambda < 0$.

$$\left(\frac{1}{2}\alpha - \frac{1}{4}b^2\beta\right)^2 - \left[\frac{b^4\beta^2}{16} - \left\{\left(\sigma - \left(\frac{\mu}{2} + \frac{3b^2\gamma}{8}\right)\right)\left(\sigma - \left(\frac{\mu}{2} + \frac{9b^2\gamma}{8}\right)\right)\right\}\right] < 0 \tag{26}$$

The effect of different physical parameters μ, the amplitude of external excitation a, nonlinearity γ, and nonlinear damping β on the frequency response curve has been drawn using Eqs. (19), (25) and (26).

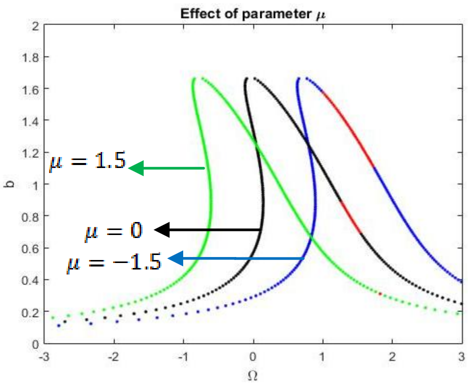

Fig. 1 Frequency response curve under the effect of parameter μ with γ = 1, a = 1, α = 0.1, β = 1

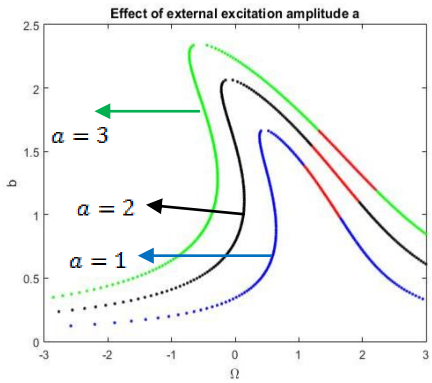

Fig. 2 Frequency response curve under the effect of amplitude of external excitation with γ = 1, μ = -1, α = 0.1, β = 1

Fig. 1 and Fig. *2* show that the amplitude varies under the frequency by varying the parameter μ and amplitude of external excitation a. It is shown that as $u \to +\infty$, the curve shifts toward the left, whereas the curve shifts the right if $u \to -\infty$. The red portion in the curve depicts the instability in the system. By increasing μ , the instability reduces. Similarly, the amplitude of the oscillation grows if the amplitude of external excitation increases. Additionally, the instability also grows as excitation amplitude a grows.

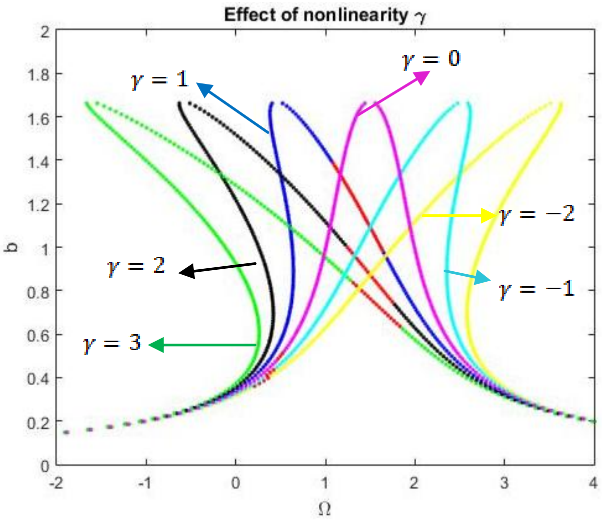

Fig. 3 Frequency response curve of the effect of nonlinearity γ with $a = 1,\ \alpha = 0.1,\ \beta = 1,\ \mu = -1$

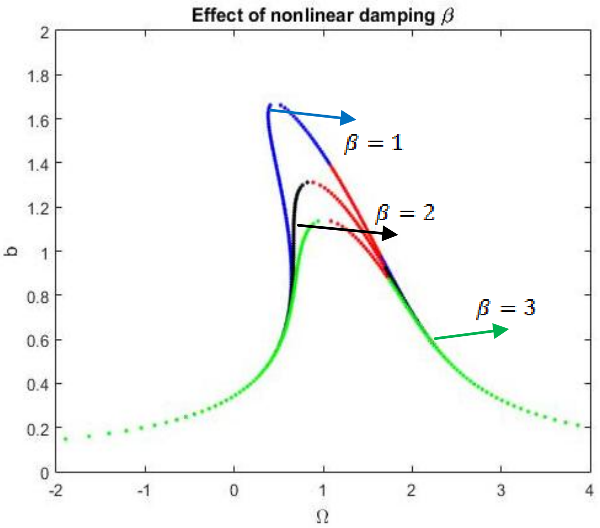

Fig. 4 Frequency response curve under the effect of nonlinear damping β with $\gamma = 1,\ a = 1,\ \alpha = 0.1,\ \mu = -1$

Figs. 3 and 4 show the variation of amplitude of oscillation versus frequency under the effects of nonlinearity and nonlinear damping, respectively. It is shown that if $\gamma \to +\infty$, the curve turns to the right, whereas the curve turns to the left if $\gamma \to -\infty$. Additionally, it is also shown that the instability in the system decreases if γ increases or decreases, that is, away from zero. Moreover, Fig. 4 shows that amplitude and instability decreases if the damping increases.

4.2. Case II: $\cos x \cong \left(1 - \frac{x^2}{2!}\right)$

By substituting the above expression for $\cos x$ and by taking $h,\ \alpha,\ \beta,\ \gamma,\ a = O(\epsilon)$ into Eq. (1), it yields:

$$\ddot{x} + x = \epsilon\left[xh\cos\omega t + (\alpha - \beta x^2)\dot{x} + \gamma x^3 + a\Omega^2\cos\Omega t - \frac{x^2}{2!}\,a\Omega^2\cos\Omega t\right] \tag{27}$$

By using Eq. (5) in (27) it follows:

$$\frac{d^2x}{ds^2} + x = \epsilon\left[\mu x + xh\cos(s) + (\alpha - \beta x^2)\frac{dx}{ds} + \gamma x^3 + a\Omega^2\cos\Omega s - \frac{x^2}{2!}\,a\Omega^2\cos\Omega s\right] \tag{28}$$

where $\omega^{-2} = 1 - \epsilon\mu$ and $\omega^{-1} = (1-\epsilon\mu)^{\frac{1}{2}}$.

By plugging Eq. (7) and (8) into Eq. (28) and equating the resulting equations for $\dot{b}$ and $\dot{\psi}$, we obtain:

$$\dot{b} = -\epsilon[\mu b\cos(s+\psi)\sin(s+\psi) + bh\cos(s)\cos(s+\psi)\sin(s+\psi) - b\alpha\sin^2(s+\psi) + b^3\beta\sin^2(s+\psi)\cos^2(s+\psi) + b^3\gamma\cos^3(s+\psi)\sin(s+\psi) + a\Omega^2\cos\Omega s\sin(s+\psi) - \frac{ab^2\Omega^2}{2}\cos\Omega s\cos^2(s+\psi)\sin(s+\psi)] \tag{29}$$

$$\dot{\psi} = -\epsilon[\mu\cos^2(s+\psi) + h\cos(s)\cos^2(s+\psi) - \alpha\sin(s+\psi)\cos(s+\psi) + b^2\beta\sin(s+\psi)\cos^3(s+\psi) + b^2\gamma\cos^4(s+\psi) + \frac{a}{b}\Omega^2\cos\Omega s\cos(s+\psi) - \frac{ab\Omega^2}{2}\cos\Omega s\cos^3(s+\psi)] \tag{30}$$

The right-hand sides of (29)-(30) are 2π periodic in s so the averaging of (29)-(30) yields.

$$\dot{b} = \frac{1}{2}\epsilon b\,\alpha - \frac{1}{8}\epsilon b^3\beta - \frac{\Omega^2 b^2}{16}\epsilon a\,\sin[(\Omega - 3) - 3\psi] \tag{31}$$

$$\dot{\psi} = -\frac{1}{2}\epsilon\mu - \frac{3}{8}\epsilon b^2\gamma + \frac{\Omega^2 b}{16}\epsilon a\,\cos[(\Omega - 3) - 3\psi] \tag{32}$$

Let us put $\Omega - 3 = \epsilon\sigma$ and $\eta = \sigma s - \psi$ in Eqs. (31) and (32) we get

$$\dot{b} = \frac{1}{2}b\alpha - \frac{1}{8}b^3\beta - \frac{9b^2}{16}a\sin\eta \tag{33}$$

$$\frac{\dot{\eta}}{3} = \frac{\sigma}{3} + \frac{1}{2}\mu + \frac{3}{8}b^2\gamma - \frac{9b}{16}a\cos\eta \tag{34}$$

For a steady state, we shall make $\dot{b}$ and $\dot{\eta}$ equal to zero into Eq. (33) and (34) it yields:

$$\frac{1}{2}b\alpha - \frac{1}{8}b^3\beta - \frac{9b^2}{16}a\sin\eta = 0 \tag{35}$$

$$\frac{\sigma}{3} + \frac{1}{2}\mu + \frac{3}{8}b^2\gamma - \frac{9b}{16}a\cos\eta = 0 \tag{36}$$

Dividing Eq. (35) by 'b', if follows

$$\frac{1}{2}\alpha - \frac{1}{8}b^2\beta - \frac{9b}{16}a\sin\eta = 0 \tag{37}$$

Now squaring Eq. (36) and (37) and adding the resulting equations, we get

$$\sigma = -\frac{3}{2}\mu - \frac{9}{8}b^2\gamma \pm \frac{3}{2}\left[\frac{81b^2a^2}{64} - \alpha^2 + \frac{1}{2}b^2\alpha\beta - \frac{1}{16}b^4\beta^2\right]^{\frac{1}{2}} \tag{38}$$

Putting $\sigma = \frac{1}{\epsilon}(\Omega - 3)$ into Eq. (39) yields:

$$\Omega = 3 + \left[-\frac{3}{2}\mu - \frac{9}{8}b^2\gamma \pm \frac{3}{2}\left[\frac{81b^2a^2}{64} - \alpha^2 + \frac{1}{2}b^2\alpha\beta - \frac{1}{16}b^4\beta^2\right]^{\frac{1}{2}}\right] \tag{39}$$

Thus, Eq. (39) is the required equation of frequency-response curve for this case, $\cos x \cong \left(1 - \frac{x^2}{2!}\right)$.

4.2.1. Stability Analysis

To examine the stability of the system, we shall find the eigenvalues of the system. To do so, we compute the Jacobean of the coupled system of ordinary differential equations (35)-(36).

$$J = \begin{pmatrix} \frac{1}{2}\propto - \frac{3}{8}b^2\beta - \frac{18b}{16}a\sin\eta & -\frac{9b^2}{16}a\cos\eta \\ \frac{6}{8}b\gamma - \frac{9}{16}a\cos\eta & \frac{9b}{16}a\sin\eta \end{pmatrix} \tag{40}$$

The critical points p of the averaged Eqs. (33) and (34) satisfy the following transcendental equations:

$$\frac{1}{2}b\propto - \frac{1}{8}b^3\beta = \frac{9b^2}{16}a\sin\eta \tag{41}$$

$$\frac{\sigma}{3} + \frac{1}{2}\mu + \frac{3}{8}b^2\gamma = \frac{9b}{16}a\cos\eta \tag{42}$$

with Eq. (41) and (42), Eq. (40) becomes:

$$J = \begin{pmatrix} -\frac{1}{2}\propto - \frac{1}{8}b^2\beta, & -\frac{b\sigma}{3} - \frac{b\mu}{2} - \frac{3b^3\gamma}{8} \\ -\frac{\sigma}{3b} - \frac{\mu}{2b} + \frac{3b\gamma}{8} & \frac{1}{2}\propto - \frac{1}{8}b^2\beta, \end{pmatrix}_p \tag{43}$$

For eigenvalues $|J - \lambda I| = 0$, where I is the identity matrix of order 2×2, thus the eigenvalues of the system are given as under:

$$\lambda_{1,2} = -\frac{1}{8}b^2\beta \pm \left[\frac{1}{4}\alpha^2 + \left\{\left(\frac{\sigma}{3} + \frac{\mu}{2} + \frac{3b^2\gamma}{8}\right)\left(\frac{\sigma}{3} + \frac{\mu}{2} - \frac{3b^2\gamma}{8}\right)\right\}\right]^{\frac{1}{2}} \tag{44}$$

The system is unstable if λ > 0 so for stable system λ < 0.

$$\left(\frac{1}{8}b^2\beta\right)^2 - \left[\frac{1}{4}\alpha^2 + \left\{\left(\frac{\sigma}{3} + \frac{\mu}{2} + \frac{3b^2\gamma}{8}\right)\left(\frac{\sigma}{3} + \frac{\mu}{2} - \frac{3b^2\gamma}{8}\right)\right\}\right] < 0 \tag{45}$$

A frequency response curve under the effect of parameter μ, amplitude of external excitation a, positive nonlinearity γ, and negative nonlinearity γ are drawn using equations (39), (44) and (45).

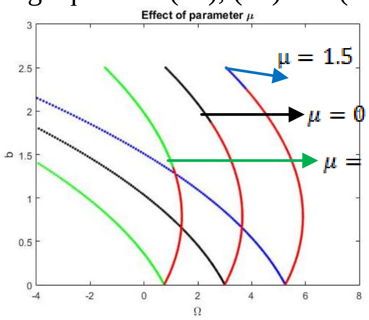

Fig. 5 Frequency response curve under the effect of parameter μ with $\gamma = 1,\ a = 1,\ \alpha = 0.01,\ \beta = 1$

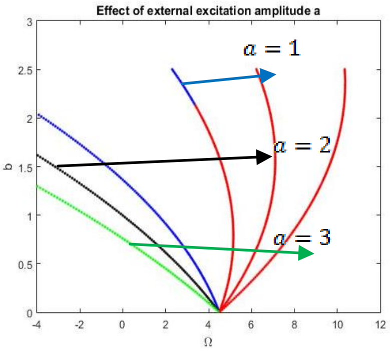

Fig. 6 Frequency response curve under the effect of external excitation a with $\gamma = 1,\ \mu = -1,\ \alpha = 0.01,\ \beta = 1$

Figs. 5 and 6 show the frequency response curve under the effects of the parameter μ and the amplitude b. It is shown that the curves move toward right as the parameter μ increases. Additionally, the instability decreases with increasing the parameter μ. It is shown that the curve stretches as the excitation amplitude b increases and the instability increases as the excitation amplitude increases.

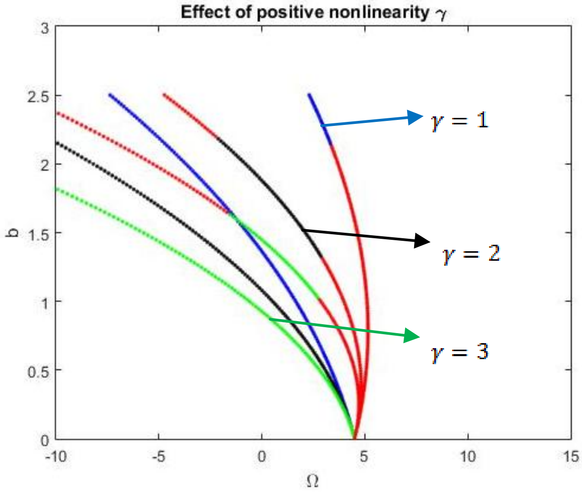

Fig. 7 Frequency response curve of the effect of nonlinearity γ with $a = 1,\ \alpha = 0.01,\ \beta = 1,\ \mu = -1$

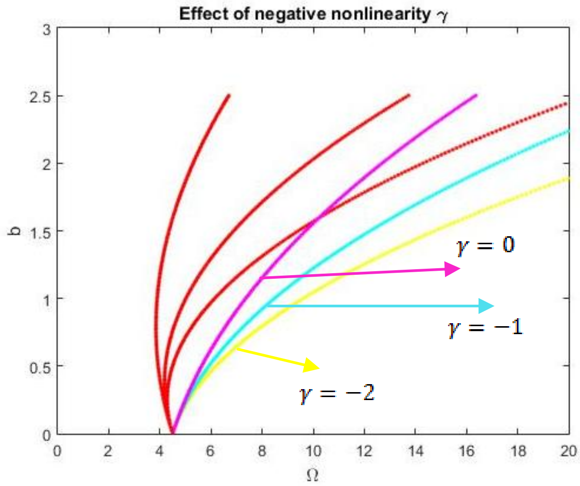

Fig. 8 Frequency response curve under the effect of nonlinearity γ with $a = 1,\ \alpha = 0.01,\ \beta = 1,\ \mu = -1$

Figs. 7 and 8 exhibit the frequency response curves under the effect of nonlinearity. For positive nonlinearity γ, it is shown that the instability regions manifold as γ increases. By decreasing γ, the instability regions decreases and the curve stretches. For negative nonlinearity γ, it is shown that the instability region decreases as the parameter γ moves toward zero.

4.3. Case III: $\cos x \cong \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)$

Substituting the expression for $\cos x$ and by taking $h,\ \alpha,\ \beta,\ \gamma,\ a = O(\epsilon)$ into Eq. (1) and adopting the similar steps as carried out for case I and Case II we get the following frequency response equation as under:

$$\Omega = 5 + \epsilon\left[-\frac{5}{2}\mu - \frac{15}{8}b^2\gamma \pm \frac{5}{2}\left[\left(\frac{25\,ab^3}{4!\times 16}\right)^2 - \epsilon^2 + \frac{1}{2}b^2\epsilon\beta - \frac{1}{16}b^4\beta^2\right]^{\frac{1}{2}}\right] \tag{46}$$

4.3.1. Stability Analysis

For stability purposes, we shall find the eigenvalues of the systems for this case, which are obtained using similar types of steps as carried out for Case I and Case II and are obtained as under

$$\lambda_{1,2} = -\left[\frac{1}{2}\alpha\right] \pm \left[\left(\alpha - \frac{1}{8}b^2\beta\right)^2 + \left\{\left(\frac{\sigma}{5} + \frac{\mu}{2} + \frac{3b^2\gamma}{8}\right)\left(\frac{3\sigma}{5} + \frac{3\mu}{2} + \frac{3b^2\gamma}{8}\right)\right\}\right]^{\frac{1}{2}} \tag{47}$$

For this case system is Instable if $\lambda > 0$, that is

$$\left(\frac{1}{2}\alpha\right)^2 - \left[\left(\alpha - \frac{1}{8}b^2\beta\right)^2 + \left\{\left(\frac{\sigma}{5} + \frac{\mu}{2} + \frac{3b^2\gamma}{8}\right)\left(\frac{3\sigma}{5} + \frac{3\mu}{2} + \frac{3b^2\gamma}{8}\right)\right\}\right] < 0 \tag{48}$$

A frequency response curve under the effect of parameter μ, amplitude of external excitation a, positive nonlinearity γ, and negative nonlinearity γ are drawn using equations (46), (47) and (48).

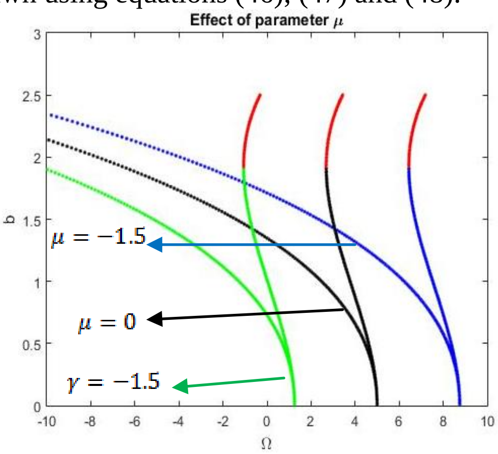

Fig. 9 Effect of frequency response curve parameter μ with γ = 1, a = 4, α = 0.00000001, β = 0.001

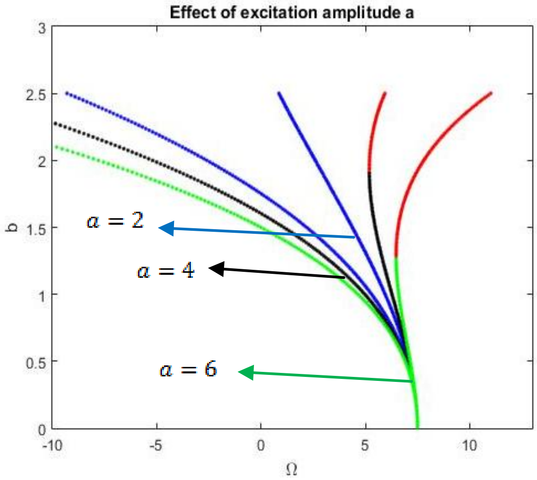

Fig. 10 Frequency response curve under the effect of amplitude of external excitation with γ = 1, μ = -1, α = 0.1, β = 1

Figs. 9 and 10 show the frequency response curve under the effect of the parameter μ and the amplitude b. It shows that the curve moves toward the left as the parameter μ increases. Further, the instability increases as the parameter μ increases. It is shown that instability increases the amplitude b increases.

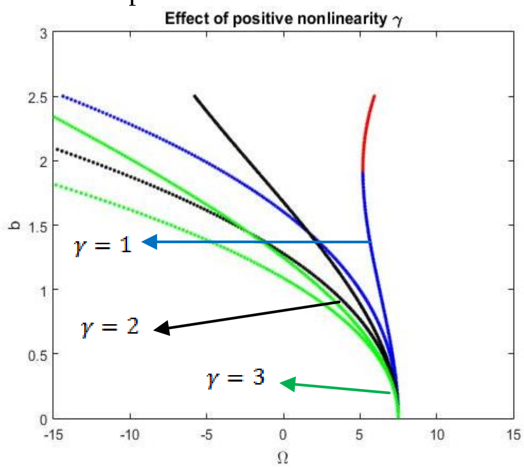

Fig. 11 Frequency response curve with nonlinearity effect γ with $a = 4,\ \alpha = 0.00000001,\ \beta = 0.001,\ \mu = -1$

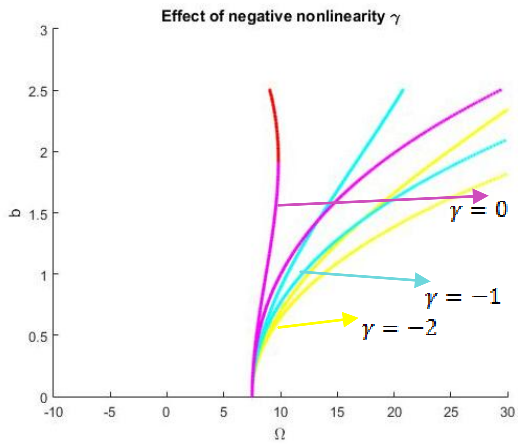

Fig. 12 Frequency response curve with nonlinearity effect γ with $a = 2,\ \alpha = 0.00000001,\ \beta = 0.001,\ \mu = -1$

Figs. 11 and 12 exhibit the frequency response curves under the effect of nonlinearity. For positive nonlinearity, it is shown that the curve moves toward right as the nonlinearity increases. Additionally, the instability decreases as the nonlinearity increases. For negative nonlinearity, it is shown that the curve moves toward the left as the nonlinearity increases in negative. Additionally, the instability decreases as the nonlinearity decreases in negative.

4.4. Case IV: $\cos x \cong \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right)$

Substitution of the above expression for $\cos x$ and by taking $h, \alpha, \beta, \gamma, a = O(\epsilon)$ into Eq. (1) and adopting the similar steps as carried out for cases I to III w get following equation of frequency response:

$$\Omega = 7 + \epsilon\left[-\frac{7}{2}\mu - \frac{21}{8}b^2\gamma \pm \frac{7}{2}\left[\left(\frac{49b^5}{6!\times 64}\right)^2 - \alpha^2 + \frac{1}{2}b^2\alpha\beta - \frac{1}{16}b^4\beta^2\right]^{\frac{1}{2}}\right] \tag{49}$$

4.4.1. Stability Analysis

For stability purposes, we shall find the eigenvalues of the systems for this case, which are obtained using similar types of steps as carried out for Case I to III and are obtained as under

$$\lambda_{1,2} = -\left[\alpha - \frac{1}{8}b^2\beta\right] \pm \left[\left(\frac{3}{2}\alpha - \frac{2}{8}b^2\beta\right)^2 + \left\{\begin{matrix}\left(\frac{\sigma}{7} + \frac{\mu}{2} + \frac{3b^2\gamma}{8}\right)\\ \left(\frac{5\sigma}{3} + \frac{5\mu}{2} + \frac{9b^2\gamma}{8}\right)\end{matrix}\right\}\right]^{\frac{1}{2}} \tag{50}$$

This is the instability if $\lambda > 0$

$$\left(\alpha - \frac{1}{8}b^2\beta\right)^2 - \left[\left(\frac{3}{2}\alpha - \frac{2}{8}b^2\beta\right)^2 + \left\{\left(\frac{\sigma}{7} + \frac{\mu}{2} + \frac{3b^2\gamma}{8}\right)\left(\frac{5\sigma}{3} + \frac{5\mu}{2} + \frac{9b^2\gamma}{8}\right)\right\}\right] < 0 \tag{51}$$

The frequency response curve under the effect of parameter μ, amplitude of external excitation a, positive nonlinearity γ, and negative nonlinearity γ are drawn by using equations (49), (50) and (51).

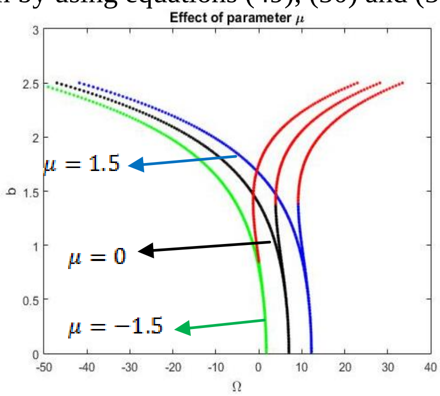

Fig. 13 Frequency response curve under the effect of parameter μ with $\gamma = 1$, $a = 100$, $\alpha = 0.00000000001$, $\beta = 0.0000001$

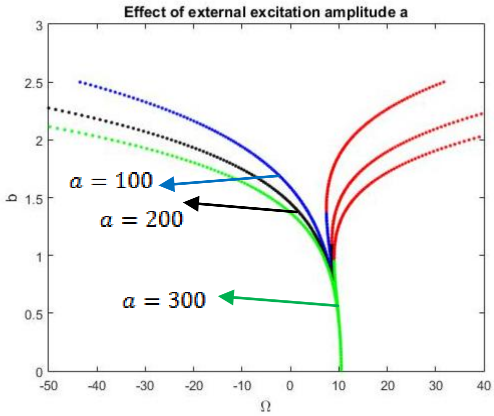

Fig. 14 Frequency response curve under the effect of amplitude of external excitation a with $\gamma = 1$, $\mu = -1$, $\alpha = 0.00000000001$, $\beta = 0.0000001$

Figs. 13 and 14 exhibit the frequency response curve under the effect of parameter μ and the amplitude b. It is shown that the curve moves left as the parameter μ increases and subsequently instability increases as μ increases. It is shown that the curve moves upward as the amplitude excitation b decreases; whereas the instability grows as the amplitude b grows.

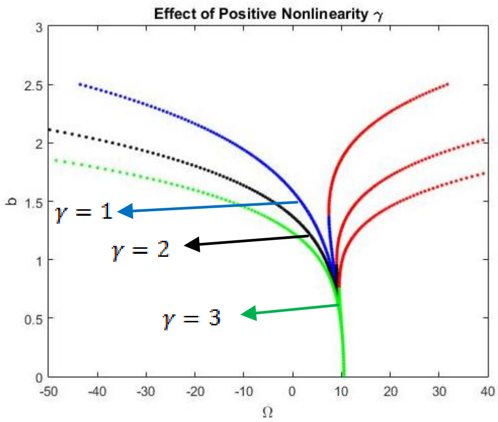

Fig. 15 Frequency response curve under the effect of nonlinearity γ with $a = 100$, $\alpha = 1$, $\beta = 0.01$, $\mu = -1$

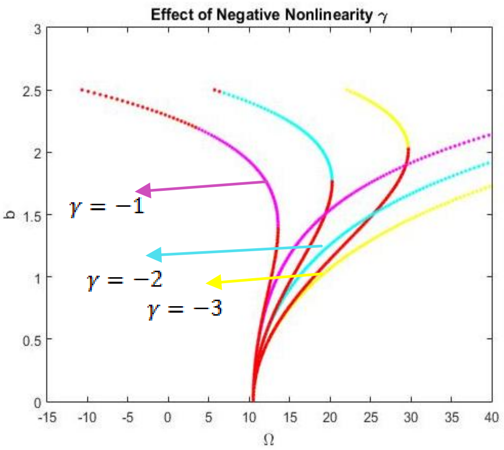

Fig. 16 Frequency response curve under the effect of nonlinearity γ with $a = 100$, $\alpha = 1$, $\beta = 0.01$, $\mu = -1$

Figs. 15 and 16 exhibit the frequency response curve under the effect of nonlinearity. For positive nonlinearity γ, it is shown that the curve moves upward as the nonlinearity γ increases and subsequently the instability region grows as the nonlinearity increases. For negative nonlinearity γ, it is shown that the curve moves left as the nonlinearity γ increases from the negative side and subsequently the instability region increases.

5. Conclusion

In this paper, the stability of the van der Pol-Mathieu-Duffing oscillator under the effects of damping nonlinearity and fast harmonic excitation is examined via Krylov-Bogoliubov averaging technique. It is found that the resonances occur at at $\Omega \approx 1,3,5,7$ depending upon the expansion of external excitation term $\cos x$. For the $\cos x \approx 1$, resonance occur at $\Omega \approx 1$. whereas for $\cos x \approx \left(1-\frac{x^2}{2!}\right)$, $\cos x \approx \left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\right)$and $\cos x \approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$ the resonance occurs at $\Omega \approx 3, 5$ and 7, respectively.

The expressions for frequency-response curves are obtained at various resonant conditions. The main finding of this study is that for the case of resonance $\Omega \approx 1$, the frequency-response curves have a classical bell shape, in this situation the system is non-chaotic. Further, it is seen that the amplitude of the oscillation grows as excitation increases, while the damping suppresses it. Moreover, the entrainment area is seen to shift toward right if nonlinearity decreases and toward left if it increases. For the case of resonances $\Omega \approx 3, 5$ and 7 the frequency-response curves first decreases to zero and then increases. It is seen that the entrainment region stretches as the amplitude of external excitation increases while compresses when excitation decreases, Furthermore, it is shown that the entrainment region shifts toward right if nonlinearity increases, while with a decrease in nonlinearity, the entrainment region shifts toward left. In the future, the averaging method can be applied to study the van der Pol-Mathieu-Duffing oscillator for external excitations other than fast harmonic excitation.

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