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Approximation Technique for Solving High-Order Non-Oscillatory Vibration Systems with Slowly Changing Coefficients Represented by Strong Non-Linearity and Multiple Integrated Roots

Nasir Uddin[1], Rezaul Karim[1], Sanjay Kumar Saha[1], Hasan Al Mamun[1], Shuvo Sarker[1], Md. Rashedul Islam[1], M. Ali Akbar[2], Pinakee Dey[1]*

[1] Department of Mathematics, Mawlana Bhashani Science and Technology University, Tangail 1902, Bangladesh

[2] Department of Applied Mathematics, Rajshai University, Bangladesh

Abstract: A formula for the asymptotic solution of an over-damped strongly nonlinear vibration system combining the extended KBM method and the harmonic balance (HB) method with slowly varying coefficients is proposed. This article aims to establish a slowly time-varying solution of an over-damped nonlinear vibration system where one eigenvalue is an integer multiple (greater than two hundred times) of the other eigenvalues. The integrated multiple eigenvalues can provide a better result than other eigenvalues for strong linearity (even if). We found solutions by considering initial conditions and comparing the percentage error of present solutions with unified solutions by using this procedure in examples. Finally, the findings are addressed, especially to improve the physical prospects and shown graphically by using Excel, Dev C++, MATHEMATICA, and MATLAB software.

Keywords: nonlinear system, over damped vibration system, slowly changing coefficients, multiple integrated roots, perturbation equations, strongly non-linear.

求解以強非線性和多重積分根為代表的緩變係數高階非振盪振動系統的逼近技術

摘要：提出了結合擴展克雷洛夫-博戈柳博夫平均法方法和緩變係數諧波平衡方法的過阻尼強非線性振動系統的漸近解公式。本文旨在建立一個過阻尼非線性振動系統的緩慢時變解，其中一個特徵值是其他特徵值的整數倍（大於二百倍）。對於強線性（即使），集成的多個特徵值可以提供比其他特徵值更好的結果。我們通過考慮初始條件並通過在示例中使用此過程將當前解決方案與統一解決方案的百分比誤差進行比較來找到解決方案。最後，研究結果得到解決，特別是改善物理前景，並使用 Excel、Dev C++、數學和软件軟件以圖形方式顯示。

关键词：非線性系統，過阻尼振動系統，緩慢變化的係數，多重積分根，微擾方程，強非線性。

1. Introduction

Many analytical methods have found limited solutions for different non-linear systems. Among them are the widely used perturbation methods, wherein a small parameter power amplifies the solution. [1–5] are important. To avoid the complexity of algebra, the perturbation method determines a low-order approximate solution. Another significant method is the harmonic balance (HB) method [6–10], which covers strongly nonlinear systems. The Krylov–Bogoliubov

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About the authors: Nasir Uddin, Rezaul Karim, Sanjay Kumar Saha, Hasan Al Mamun, Shuvo Sarker, Md. Rashedul Islam, Department of Mathematics, Mawlana Bhashani Science and Technology University, Tangail, Bangladesh; M. Ali Akbar, Department of Applied Mathematics, Rajshai University, Bangladesh; Pinakee Dey, Department of Mathematics, Mawlana Bhashani Science and Technology University, Tangail, Bangladesh

Corresponding author Pinakee Dey, pinakeedey68@gmail.com

averaging method (KBM) [11–13] is one of the the above-mentioned methods that is well-known for analyzing the theory of nonlinear oscillations. Initially, [3] developed this method to obtain periodic solution of nonlinear second-order differential systems. Letter, the method was amplified and verified mathematically by [4]. To process a damped oscillatory, [14] extended this method by using a strong linear damping force. Further, to advance an over-damped nonlinear system, [15]] encompassed this method. [17–19] examined over-damped nonlinear systems and found approximate solutions of *Duffing*'s equation, when the number of undisturbed equation roots is more than one times. Again, [18] proposed a unified method to solve an *n*-th order differential system (autonomous) and characterized by using constant coefficient and slowly varying coefficient-based oscillatory, damped oscillatory and non-oscillatory processes. [20] extended KBM method by using the slowly and periodically changing coefficients-based on underdamping, damping, and overdamping vibrating systems. [20], [21] introduced a damped forced nonlinear vibrating system with varying coefficients. At present, [22] also extended the method by using slowly and periodically changing coefficients-based damped and damped forced vibrating systems with strong non-linearity. In another recent paper, [23] finds approximate solutions to over damped nonlinear differential systems based on the extended KBM method, where one eigenvalue is a multiple (ten times) of the other eigenvalues. The purpose of this article is to find a solution for a slowly changing over-damped nonlinear vibration system where one eigenvalue is an integral multiple (more than 200 times) of the other eigenvalues.

2. Methodology

Let us consider nonlinear differential systems governed by

$$\ddot{x} + 2\xi_1(\tau)\dot{x} + (\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 \sin 2\tau)x = -\varepsilon f(x,\dot{x},\tau),\ \tau = \varepsilon t \tag{1}$$

where the over-dots indicate differentiation with respect to t, ε an inconsiderable parameter, $\zeta_1 = \zeta_2 = 0(\varepsilon) = \zeta_3$, $\tau = \varepsilon t$ slowly varying time, $\xi(\tau) \geq 0$, f a nonlinear function. Since their time derivatives are proportional to ε, the coefficients in equation (1) change slowly. We set, $\omega^2(\tau) = (\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 \sin 2\tau)$ where $\omega(\tau)$ is known as the internal frequency. Setting $\varepsilon = 0$ and $\tau = \tau_0 =$ constant, in Eq. (1). Eq. (1) has two eigenvalues, $\lambda_1(\tau_0)$ and $\lambda_2(\tau_0)$ are constants, but when $\varepsilon \neq 0$, $\lambda_1(\tau_0)$ and $\lambda_2(\tau_0)$ change slowly in that their time derivatives. We may consider that $|\lambda_2(\tau_0)| >> |\lambda_1(\tau_0)|$. The unperturbed solution of equation (1) is

$$x(t,0) = x_0 e^{\lambda_1(\tau_0)t} + y_0 e^{\lambda_2(\tau_0)t}, \tag{2}$$

When $\varepsilon \neq 0$, we propound an asymptotic solution for equation (1)

$$x(t,\varepsilon) = x_{1,0}(t,\tau) + y_{1,0}(t,\tau) + \varepsilon u_1(x,y,t,\tau) + \varepsilon^2 u_2(x,y,t,\tau) + \ldots, \tag{3}$$

where x_1 and y_1 satisfies first-order differential equations

$$\dot{x}_1 = \lambda_1(\tau)x_1 + \varepsilon X_1(x_1,y_1,\tau) + \varepsilon^2 X_2(x_1,y_1,\tau)$$
$$\dot{y}_1 = \lambda_2(\tau)y_1 + \varepsilon Y_1(x_1,y_1,\tau) + \varepsilon^2 Y_2(x_1,y_1,\tau) \tag{4}$$

Limited to the first few terms only, 1, 2…*m*, a series of expansions of (3) and (4), we determine the function $u_1, u_2, \ldots,$ and $X_1, X_2, \ldots, Y_1, Y_2, \ldots,$ such that $x_1(t,\tau)$ and $y_1(t,\tau)$ appears in (3) and (4) and fulfills the given differential system (1) with precision ε^{m+1} [15] to determine these unknown features. The function does not contain terms that include; $\varepsilon^{\lambda_j t}$, $j = 1,2$, these are because they are incorporated in the series expansion (3) at the time of ordering ε^0. As these unknown functions are determined, the functions $u_1, u_2, \ldots$ not included secular-type terms te^{-t} [15, 18, 19, 20, 24–26] Differentiate x (t, ε) twice with respect to t and substitute the derivatives ẍ and x in the original equation (1) to equalize the coefficients of the equal harmonics. We get

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1) X_1 + (\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2) Y_1 + \lambda_1' x_1 + \lambda_2' y_1 + (\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1)(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2) u_1 = -f^{(0)}(x_1,y_1,\tau), \tag{5}$$

where

$\lambda_1' = d\lambda_1/d\tau$, $\lambda_2' = d\lambda_2/d\tau$, $\Omega x_1 = \partial/\partial x_1$, $\Omega y_1 = \partial/\partial y_1$, $f^{(0)} = f(x_0,\dot{x}_0,\tau)$

Here it is assumed that $f^{(0)}$ can be expanded in the Fourier series as

$$f^{(0)} = \sum_{r_1,r_2=0}^{\infty} F_{r_1,r_2}(\tau) x_1^{r_1} y_1^{r_2} \tag{6}$$

The formulae are obtained by equating the coefficients of equal harmonic terms on both sides. To get the solution (1) with overdamping, apply a constraint that $u_1 \cdots$ excludes the terms

$x^{i_1} y^{i_2}, i_1\lambda_1 + i_2\lambda_2 < (i_1 + i_2)\xi(\tau_0), i_1, i_2 = 0,1,2.$

This assumption confirms that there are no terms of the secular type $te^{-\lambda_1 t}$ [15, 16, 20, 24–26], assuming that this research can found the unknown function u_1 and X_1, Y_1, so this completes the evaluation of the first order solution in (1).

2.1. Example

Consider a nonlinear autonomous vibrating system governed by

$$\ddot{x} + 2\xi_1(\tau)\dot{x} + (\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 \sin 2\tau)x = -\varepsilon x^3 \tag{7}$$

Eq. (7) has two eigenvalues. $\lambda_j(\tau_0), j = 1,2$ Here, $x_0 = x_1 + y_2$ $f^{(0)} = -(x_1^3 + 3x_1^2 y_2 + 3x_1 y_2^2 + y_2^3)$ and $\Omega x_1 = \partial/\partial x_1, \Omega y_1 = \partial/\partial y_1$. Now, by replacing with the value of $f^{(0)}$ Eq. (5), we obtain

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2) X_1 + (\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1) Y_1 + \lambda_1' x_1 + \lambda_2' y_1 = -(x_1^3 + 3x_1^2 y_1) \quad (8)$$

and

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1)(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2) u_1 = -(3x_1 y_1^2 + y_1^3) \quad (9)$$

The particular solution of (9) is

$$u_1 = \alpha_1 x_1 y_1^2 + \alpha_2 y_1^3 \quad (10)$$

where

$$\alpha_1 = -3/2\lambda_2(\lambda_1 + \lambda_2), \ \alpha_2 = -1/2\lambda_2(3\lambda_2 - \lambda_1)$$

Now we solve the two functions of (7) X and Y (described in the methodology). The specific solutions are

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2) X_1 + \lambda_1' x_1 = -x_1^3 \quad (11)$$

and

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1) Y_1 + \lambda_2' y_1 = -3x_1^2 y_1 \quad (12)$$

To the particular solution of (11)-(12) and replacing the functional values of, X_1 Y_1 with (4) and rearrange, we obtain

$$\dot{x}_1 = \lambda_1 x_1 + \varepsilon(\lambda_1' x_1 \beta_1 + \beta_2 x_1^3) \quad (13)$$

and $$\dot{y}_1 = \lambda_2 y_1 + \varepsilon(\lambda_2' y_1 \chi_1 + \chi_2 {x_1}^2 y_1) \quad (14)$$

where

$$\beta_1 = -1/(\lambda_1 - \lambda_2), \ \beta_2 = -1/(3\lambda_1 - \lambda_2),$$
$$\chi_1 = 1/(\lambda_1 - \lambda_2), \quad \chi_2 = -3/(\lambda_1 + \lambda_2)$$

Now we must solve the equation. (13) and equation. (14) for an x_1 y_1; but Eq. (13) and Eq. (14) have an exact solution or not. In most of the cases (i.e., over damped or critically damped), we can unable to find an exact solution of equation. (4) when the nonlinear equation has a physically powerful linear damping force [16, 20, 24, 26–30]. For over damped system, [15] replace the terms with a small parameter ε, through their respective unperturbed values (i.e., $x_1(t), y_1(t)$ by $x_0 e^{\lambda_1(\tau_0)t}andy_0 e^{\lambda_2(\tau_0)t}$), since x together with all $x_1(t), y_1(t)$ die out quick. Within this time interval, the difference between $x_1(t), y_1(t)$ and$x_0 e^{\lambda_1(\tau_0)t}, y_0 e^{\lambda_2(\tau_0)t}$ occurs in the order of ε only. Yet, because of motions with little damping or without damping, this is certainly off-base. Here, x and $x_1(t), y_1(t)$ die out occurs in more than an order ε. In this article we used Runge-Kutta method (4th order).

Hence, the first order solution of equation (7) is

$$x(t,\varepsilon) = x_1 + y_1 + \varepsilon u_1, \quad (15)$$

where x_1 and y_2 are given by (13) and (14), and u_1 is given by (10).

2.2. Another Formation

We may consider another formation. Therefore, we choose

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2) X_1 + (\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1) Y_1 + \lambda_1' x_1 + \lambda_2' y_1 = -(x_1^3 + 3x_1^2 y_1 + {y_1}^3) \quad (16)$$

and

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1)(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2) u_1 = -3x_1 y_1^2 \quad (17)$$

The particular solution of (17) is

$$u_1 = \alpha_1 x_1 y_1^2 \quad (18)$$

where $\alpha_1 = -3/2\lambda_2(\lambda_1 + \lambda_2)$

Now we solve the two functions of (16) X and Y (described in methodology).

The particular solutions are

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_2) X_1 + \lambda_1' x_1 = -x_1^3 \quad (19)$$

and

$$(\lambda_1 x_1 \Omega x_1 + \lambda_2 y_1 \Omega y_1 - \lambda_1) Y_1 + \lambda_2' y_1 = -3x_1^2 y_1 - {y_1}^3 \quad (20)$$

By the particular solution of (19)-(20) and substitution of functional values for X_1 Y_1 into (4) and rearrange, we obtain

$$\dot{x}_1 = \lambda_1 x_1 + \varepsilon(\lambda_1' x_1 \beta_1 + \beta_2 x_1^3) \quad (21)$$

and

$$\dot{y}_1 = \lambda_2 y_1 + \varepsilon(\lambda_2' y_1 \chi_1 + \chi_2 {x_1}^2 y_1 + \chi_3 {y_1}^3) \quad (22)$$

where

$$\beta_1 = -1/(\lambda_1 - \lambda_2), \beta_2 = -1/(3\lambda_1 - \lambda_2),$$
$$\chi_1 = 1/(\lambda_1 - \lambda_2), \chi_2 = -3/(\lambda_1 + \lambda_2), \chi_3 = -1/(3\lambda_2 - \lambda_1)$$

Consequently, the first order solution of equation (16) is

$$x(t,\varepsilon) = x_1 + y_1 + \varepsilon u_1, \quad (23)$$

where x_1 and y_2 are given by (21) and (22), and u_1 is given by (18).

2.3. Third-Order Nonlinear System

Consider nonlinear autonomous third order differential systems

$$\dddot{x} + \xi_1(\tau)\ddot{x} + \xi_2 \dot{x} + (\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3 \sin 2\tau)x = -\varepsilon f(x, \dot{x}, \ddot{x}, \tau,), \quad (24)$$

Putting $\varepsilon = 0$ and $\tau = \tau_0$ = constant, in Eq. (24), we get the non-perturbative solution of (1) in the form

$$x(t,0) = x_0 e^{\lambda_1(\tau_0)t} + y_0 e^{\lambda_2(\tau_0)t} + z_0 e^{\lambda_3(\tau_0)t}) \quad (25)$$

Let Eq. (24) has three eigenvalues, $\lambda_1(\tau_0), \lambda_2(\tau_0)$ *and* $\lambda_3(\tau_0))$ are constants.

Using Eq. (2) and Eq. (24), we get

$$(\lambda_1 x \Omega x + \lambda_2 y \Omega y + \lambda_3 z \Omega z - \lambda_2)(\lambda_1 x \Omega x + \lambda_2 y \Omega y + \lambda_3 z \Omega z - \lambda_3) X + (\lambda_1 x \Omega x + \lambda_2 y \Omega y + \lambda_3 z \Omega z - \lambda_1)(\lambda_1 x \Omega x + \lambda_2 y \Omega y + \lambda_3 z \Omega z - \lambda_3) Y + (\lambda_1 x \Omega x + \lambda_2 y \Omega y + \lambda_3 z \Omega z - \lambda_1)(\lambda_1 x \Omega x + \lambda_2 y \Omega y + \lambda_3 z \Omega z - \lambda_2) Z + \acute{\lambda}_1 x + \acute{\lambda}_2 y + \acute{\lambda}_3 z = -(3yz^2 + z^3 + y^3 + 3y^2 z + 6xyz + 3x^2 z) \quad (26)$$

and

$$(\lambda_1 x\Omega x+\lambda_2 y\Omega y+\lambda_3 z\Omega z-\lambda_1)(\lambda_1 x\Omega x+\lambda_2 y\Omega y+\lambda_3 z\Omega z-\lambda_2)(\lambda_1 x\Omega x+\lambda_2 y\Omega y+\lambda_3 z\Omega z-\lambda_3)u_1 = -(x^3+3x^2y+3x^2z+3xy^2) \quad (27)$$

The particular solution of (27) is

$$u_1=\alpha_1 x^3+\alpha_2 3x^2y+\alpha_3 3x^2 z+\alpha_4 3xy^2) \quad (28)$$

where

$$\alpha_1=-1/2\lambda_1(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)$$
$$\alpha_2=-3/(\lambda_1+\lambda_2+\lambda_3)(2\lambda_1+\lambda_3)(2\lambda_1-\lambda_2)$$
$$\alpha_3=-3/(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)(2\lambda_1)$$
$$\alpha_4=-3/(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2)$$

Now, we must solve the three functions of equation (26) $-X$, Y and Z.

The particular solutions are

$$(\lambda_1 x\Omega x+\lambda_2 y\Omega y+\lambda_3 z\Omega z-\lambda_2)(\lambda_1 x\Omega x+\lambda_2 y\Omega y+\lambda_3 z\Omega z-\lambda_3)X+\acute{\lambda}_1 x=-(3yz^2+z^3)$$
$$(\lambda_1 x\Omega x+\lambda_2 y\Omega y+\lambda_3 z\Omega z-\lambda_1)(\lambda_1 x\Omega x+\lambda_2 y\Omega y+\lambda_3 z\Omega z-\lambda_3)Y+\acute{\lambda}_2 y=-(y^3+3y^2z)$$
$$(\lambda_1 x\Omega x+\lambda_2 y\Omega y+\lambda_3 z\Omega z-\lambda_1)(\lambda_1 x\Omega x+\lambda_2 y\Omega y+\lambda_3 z\Omega z-\lambda_2)Z+\acute{\lambda}_3 z=-(6xyz+3x^2z) \quad (29)$$

By the particular solution of (29) and substitution of functional values for, X Y and Z into (4) and rearrange, we obtain

$$\dot{x}=\lambda_1 x+\epsilon(\acute{\lambda}_1 x\beta_1+\beta_2 yz^2+\beta_3 z^3)$$
$$\dot{y}=\lambda_2 y+\epsilon(\acute{\lambda}_2 y\chi_1+\chi_2 y^3+\chi_3 y^3 z) \quad (30)$$
$$\dot{z}=\lambda_3 z+\epsilon(\acute{\lambda}_3 z\delta_1+\delta_2 xyz+\delta_3 x^2 z)$$

where

$$\beta_1=-1/(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)$$
$$\beta_2=-3/2\lambda_3(2\lambda_1+\lambda_3)(\lambda_2+\lambda_3),$$
$$\beta_3=-3/2\lambda_3(3\lambda_3-\lambda_2)$$
$$\chi_1=-1/(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)$$
$$\chi_2=-1/2\lambda_3(2\lambda_1-\lambda_1)(3\lambda_2-\lambda_3)$$
$$\chi_3=-3/2\lambda_2(2\lambda_2-\lambda_{3}{}_{-}\lambda_1)$$
$$\delta_1=-1/(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)$$
$$\delta_2=-6/(\lambda_2+\lambda_3)(\lambda_1+\lambda_3)$$
$$\delta_3=-3/2\lambda_3(2\lambda_3-\lambda_2+\lambda_1)$$

Therefore, the 1st order solution of equation (24) is

$$x(t,0)=x+y+z+\varepsilon u_1), \quad (31)$$

where x y and z are given by (30) and u_1 is given by (28).

3. Results and Discussion

On the basis of extended KBM and HB methods, where the coefficients change slowly, an asymptotic solution of the overdamped nonlinear vibration system is obtained. Solutions are determined based on techniques that provide better results for strong nonlinearities. To verify the accuracy of the approximate solution obtained by the perturbation method, we compare the approximate solution with the numerical one (we consider it accurate). As for such a comparison, the extended KBM method presented and the HB method in this article [16, 20, 24, 26–30]. In this article, we compared the perturbed solutions (15) and (23) obtained using the Runge-Kutta method (4th order).

First, x is calculated according to (15) with the initial conditions $x(0)=1.000, \dot{x}(0)=0.000$ or $x_1=1.0000, y_1=-\dot{0}.169064$ for $\varepsilon=1.1, \lambda_1=-.03,\ \lambda_2=-8.$ $\omega=\omega_0\sqrt{(\zeta_1^2+\zeta_2\cos\tau+\zeta_3^2 sin2\tau)};$ (i) $x(t)$ has been computed by unified solutions [29] (A.9) with initial conditions $x(0)=1.000,\ \dot{x}(0)=0.000$ or $x_1=3.317741, y_1=-\dot{0}.238597$ for $\varepsilon=1.1, \lambda_1=-.03,\ \lambda_2=-8$ and $\omega=\omega_0\sqrt{(\zeta_1^2+\zeta_2\cos\tau+\zeta_3^2 sin2\tau)}$.

Second, $x(t)$ has been computed by unified solutions [16] (B.17) with initial conditions $x(0)=1.000, \dot{x}(0)=0.000$ or $x_1=3.317741, y_1=-\dot{1}3.0815$ for $\varepsilon=1.1,\ \lambda_1=-.03,\ \lambda_2=-8$ and

$$\omega=\omega_0\sqrt{(\zeta_1^2+\zeta_2\cos\tau+\zeta_3^2 sin2\tau)}$$

The relevant numerical solutions are calculated using the fourth-order Runge-Kutta method and are given in the second column of Table 1. The solutions are different values of x shown in the third column of Table 1. All the results are presented in Table 1. Percent errors were calculated and shown in the fourth, sixth and eighth columns of Table 1. For strong nonlinearity, the percentage error of (15) is less than 1% and a eigenvalue is a multiple (more than two hundred times) of another eigenvalue, while the percentage errors of unified solutions [30] (A.9) and the percentage of errors of unified solutions ([15, 13]) (B.17) are more than 1%. Also results are shown in Fig. 1(A), unified results [29] (A.9) in Fig. 1(B) and unified results [16] (B.17) in Fig. 1(C).

Table 1 The results

t	x_{Exact}	$x_{present\ method}$	$Er\%$	x_{Alam}	$Er\%$	x_{Murty}	$Er\%$
0	1	1	0	3.317741	0	3.317741	0
.1	0.983242	0.983462	0.02237	3.29398	-4.45337	3.29398	-29.9695
10	0.417951	0.421336	-0.8034	1.683139	-71.1043	1.683139	-73.397
20	0.264757	0.266549	-0.6723	0.931933	-68.4203	0.931933	-70.1546
30	0.183137	0.18425	-0.60407	0.562339	-64.335	0.562339	-66.0125
40	0.131144	0.131894	-0.56864	0.364528	-60.8571	0.364528	-62.5639
50	0.095451	0.09598	-0.55116	0.248882	-58.4052	0.248882	-60.1494
70	0.051628	0.051906	-0.53558	0.126714	-55.922	0.126714	-57.7126

Continuation of Table 1							
90	0.028213	0.028362	-0.52535	0.067907	-55.0871	0.067907	-56.894
100	0.020883	0.020993	-0.52398	0.050071	-54.92	0.050071	-56.7294

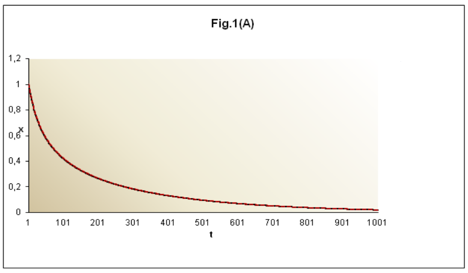

Fig. 1(A) Present solution (15) (dotted line) related numerical solution (solid line) they are drawn the initial conditions $x(0) = 1.000, \dot{x}(0) = 0.000$ or $x_1 = 1.0000, y_1 = -0.169064$ for $\varepsilon = 1.1,\ \omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 \sin 2\tau)}$. $\lambda_1 = -.03,\ \lambda_2 = -8$

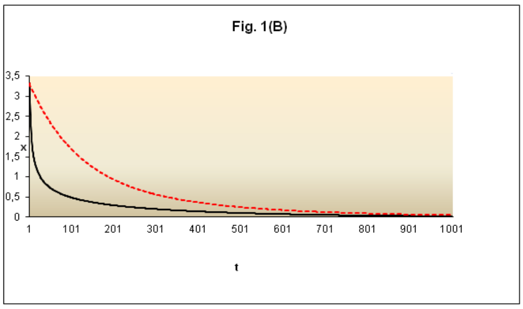

Fig. 1(B) Unified solutions [29] (A.7) (dotted line) with related numerical solution (solid line) are plotted with initial conditions $x(0) = 1.000, \dot{x}(0) = 0.000$ or $x_1 = 3.317741, y_1 = -0.238597$ for $\varepsilon = 1.1, \lambda_1 = -.03,\ \lambda_2 = -8$ and $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 \sin 2\tau)}$

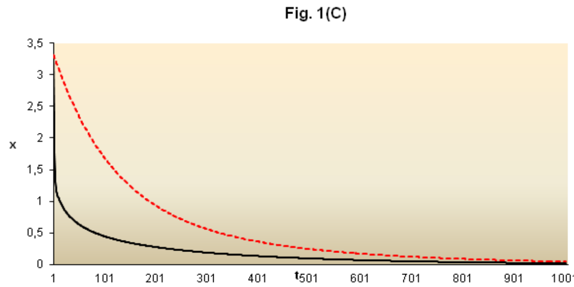

Fig. 1(C) Unified solutions [16] (B.10) (dotted line) with corresponding numerical solution (solid line) are placed with initial conditions $x(0) = 1.000$, or $\dot{x}(0) = 0.000 x_1 = 3.317741, y_1 = -13.0815$ for $\varepsilon = 1.1$ $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 \sin 2\tau)}, \lambda_1 = -.03,\ \lambda_2 = -8$

Next, we have computed by (15) and (i) unified solutions [30] (A.9) (ii) unified solutions [16] (B.17) for $\varepsilon = 1.3$. The corresponding numerical solutions have been found and the percentages of errors have been computed. The results are given in Table 2. The percentage errors of (14) are less than 1% and the percentage errors of unified results [29] (A.9) and the percentage errors of unified results [16] (B.17) are greater than 1%. The results are given in Table 2. Also results are shown in Fig. 2(A), unified solutions [30] (A.9) in Fig. 2(B) and unified solutions [16] (B.17) in Fig. (C).

Table 2 The results

t	x_{Exact}	$x_{present\ method}$	$Er\%$	x_{Alam}	$Er\%$	x_{Murty}	$Er\%$
0	1	1	0	3.739149	0	3.739149	0
1	0.980752	0.981059	0.031303	3.711612	-6.61661	3.711612	-32.8153
10	0.393852	0.397592	0.949595	1.854471	-75.9282	1.854471	-77.929
20	0.24784	0.249763	0.775904	1.001591	-73.0104	1.001591	-74.5683
30	0.171044	0.172228	0.69222	0.590661	-68.8044	0.590661	-70.3481
40	0.12236	0.123155	0.649722	0.376043	-65.1359	0.376043	-66.7363
50	0.089013	0.089572	0.627998	0.253564	-62.4856	0.253564	-64.1416
70	0.048127	0.04842	0.608806	0.127488	-59.7429	0.127488	-61.4646
90	0.026296	0.026454	0.600852	0.068035	-58.8065	0.068035	-60.5527
100	0.019464	0.01958	0.595972	0.050123	-58.6178	0.050123	-60.3675

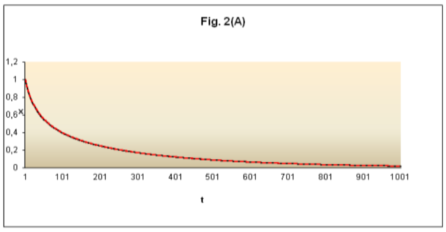

Fig. 2(A) Present solution (15) (dotted line) with similar numerical solution (solid line) they are drawn with initial situations $x(0) = 1.000$, or $\dot{x}(0) = 0.000 x_1 = 1.0000,\ y_1 = -0.194349$ for, $\varepsilon = 1.3, \omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 \sin 2\tau)}, \lambda_1 = -.03,\ \lambda_2 = -8$

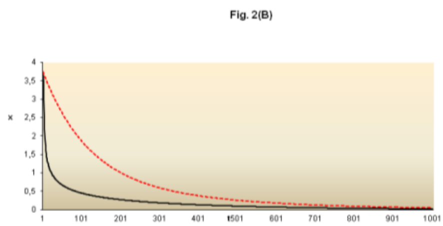

Fig. 2(B) Unified solutions [29] (A.7) (dotted line) with similar solution (solid line) are plotted with initial situation conditions $x(0) = 1.000$, or $\dot{x}(0) = 0.000 x_1 = 3.739149, y_1 = -0.276523$ for $\varepsilon = 1.3$ $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 \sin 2\tau)}, \lambda_1 = -.03,\ \lambda_2 = -8$

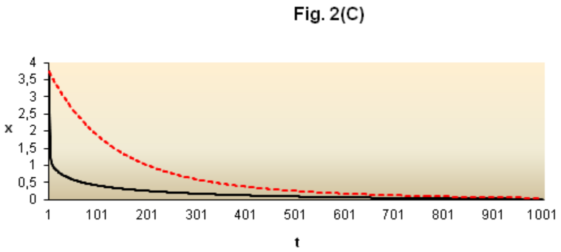

Fig. 2(C) Unified solutions [16] (B.10) (dotted line) similar numerical solutions (solid line) are plotted with initial situations $x(0) = 1.000$, or $\dot{x}(0) = 0.000$ $x_1 = 3.739149, y_1 = -15.45449$ for $\varepsilon = 1.3$

$\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2\cos\tau + \zeta_3^2 sin2\tau)}$, $\lambda_1 = -.03,\ \lambda_2 = -8$

Similarly, we have computed by (23) another set of initial conditions. The corresponding numerical solutions have been found and error percentages were calculated. The results are given in Table 3. Also results are shown in Fig. 3(A), unified solutions [29] (A.9) in Fig. 3(B) and unified results [16] (B.17) in Fig. 3(C).

Table 3 The results

t	x_{Exact}	$x_{present\ method}$	$Er\%$	x_{Alam}	$Er\%$	x_{Murty}	$Er\%$
0	1	1	0	3.107038	0	3.107038	0
.1	0.984793	0.984951	-0.01604	3.085164	-3.55589	3.085164	-28.5397
10	0.425943	0.428608	-0.62178	1.597474	-68.1076	1.597474	-70.707
20	0.262188	0.2636	-0.53566	0.897104	-65.6208	0.897104	-67.5638
30	0.174792	0.175653	-0.49017	0.548178	-61.6541	0.548178	-63.5113
40	0.120086	0.120652	-0.46912	0.358771	-58.3141	0.358771	-60.1866
50	0.083602	0.083988	-0.45959	0.246541	-55.987	0.246541	-57.8885
70	0.041151	0.041337	-0.44996	0.126327	-53.656	0.126327	-55.5954
90	0.020391	0.020482	-0.44429	0.067843	-52.878	0.067843	-54.831
100	0.014364	0.014428	-0.44358	0.050045	-52.7225	0.050045	-54.6788

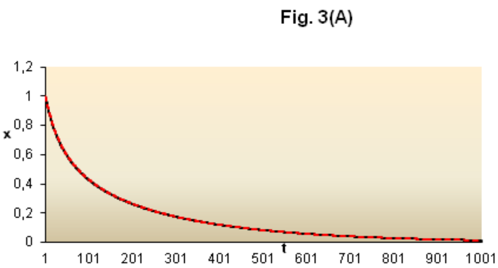

Fig. 3(A) Present solution (23) (dotted line) similar numerical solution (solid line) they are drawn with initial situations $x(0) = 1.000$, or $\dot{x}(0) = 0.000$ $x_1 = 1.0000, y_1 = -0.153413$ for $\varepsilon = 1.0$ $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2\cos\tau + \zeta_3^2 sin2\tau)}$. $\lambda_1 = -.035,\ \lambda_2 = -8.55$

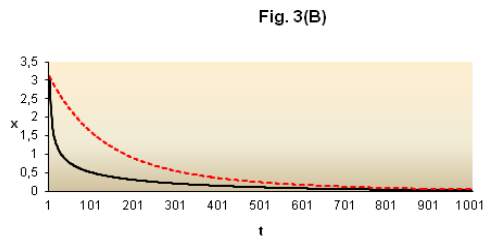

Fig. 3(B) Unified solutions [29] (A.7) (dotted line) with similar numerical solution (solid line) are plotted with initial situations $x(0) = 1.000$, or $\dot{x}(0) = 0.000$ $x_1 = 3.107038, y_1 = -0.219633$ for $\varepsilon = 1.0$ $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2\cos\tau + \zeta_3^2 sin2\tau)}$, $\lambda_1 = -.035,\ \lambda_2 = -8.55$

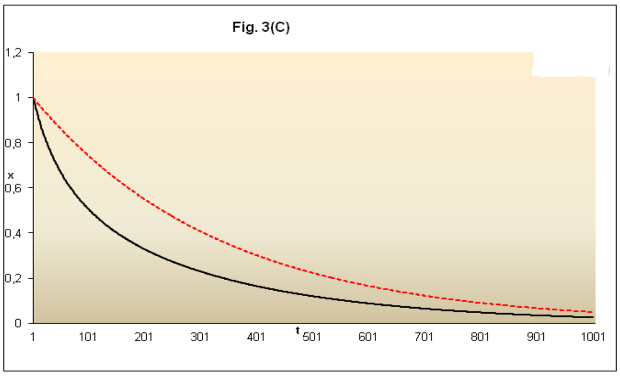

Fig. 3(C) Unified solutions [16] (B.10) (dotted line) with similar numerical solution (solid line) are plotted with initial situations $x(0) = 1.000$, or $\dot{x}(0) = 0.000$ $x_1 = 3.107038,\ y_1 = -11.8950$ for $\varepsilon = 1.0$ $\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2\cos\tau + \zeta_3^2 sin2\tau)}$, $\lambda_1 = -.035,\ \lambda_2 = -8.55$

Comparably, we have computed by (23) (i) unified solutions [29] (A.9) (ii) and unified solutions [16] (B.17) for $\varepsilon = 1.2$. The corresponding numerical solutions have been found and percentage errors have been calculated. The results are given in Table 4. Also results are shown in Fig. 4(A), unified solutions [30] (A.9) in Fig. 4(B) and unified results [16] (B.17) in Fig. 4(C).

Table 4 The results

t	x_{Exact}	$x_{present\ method}$	$Er\%$	x_{Alam}	$Er\%$	x_{Murty}	$Er\%$
0	1	1	0	3.528445	0	3.528445	0
1	0.982462	0.982688	-0.023	3.502796	-5.47277	3.502796	-31.389
10	0.400889	0.403913	-0.74868	1.768805	-73.6846	1.768805	-75.7822
20	0.244937	0.24649	-0.63005	0.966762	-70.8615	0.966762	-72.4699
30	0.162886	0.163825	-0.57317	0.5765	-66.6994	0.5765	-68.2737

Continuation of Table 4

40	0.11179	0.112405	-0.54713	0.370286	-63.1136	0.370286	-64.7308
50	0.077789	0.078207	-0.53448	0.251223	-60.5546	0.251223	-62.2176
70	0.038276	0.038478	-0.52498	0.127101	-57.935	0.127101	-59.6526
90	0.018965	0.019064	-0.5193	0.067971	-57.0479	0.067971	-58.7854
100	0.01336	0.013429	-0.51381	0.050097	-56.8697	0.050097	-58.6103

Fig. 4(A)

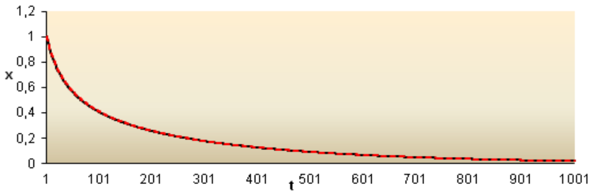

Fig. 4(A) Present solution (23) (dotted line) similar numerical solution (solid line) they are drawn with initial situations $x(0) = 1.000$, or

$x(0) = 0.000 x_1 = 1.0000,\ y_1 = -0.177096$ for $\varepsilon = 1.2$,

$\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 sin2\tau)},\ \lambda_1 = -.035,\ \lambda_2 = -8.55$

Fig. 4(B)

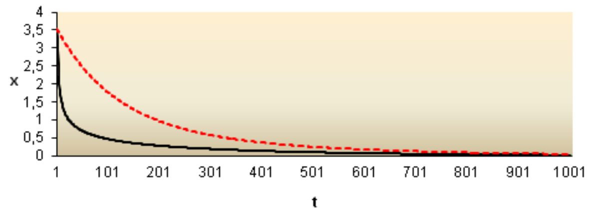

Fig. 4(B) Unified solutions [30] (A.7) (dotted line) with similar numerical solutions (solid line) are plotted with initial situations $x(0) = 1.000$, or

$x(0) = 0.000\ x_1 = 3.528445,\ y_1 = -0.257560$ for $= 1.2$

$\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 sin2\tau)},\ \lambda_1 = -.035,\ \lambda_2 = -8.55$

Fig. 4(C)

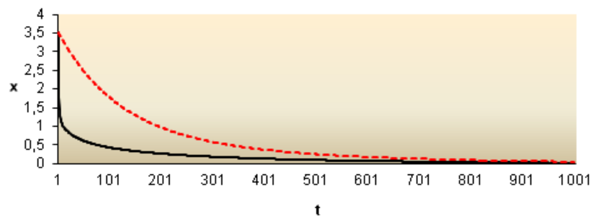

Fig. 4(C) Unified solutions [16] (B.10) (dotted line) with similar numerical solution (solid line) are plotted with initial situations $x(0) = 1.000$, or

$x(0) = 0.000 x_1 = 3.528445,\ y_1 = -14.26800$ for $\varepsilon = 1.2$

$\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 sin2\tau)},\ \lambda_1 = -.035,\ \lambda_2 = -8.55$

Finally, we have computed by (35) and unified solutions. The corresponding numerical solution is also computed by *Runge-Kutta* fourth-order method. All the results are shown in Fig. 5(A), Fig. 5(B), Fig. 6(A) and Fig. 6(B).

Fig. 5(A)

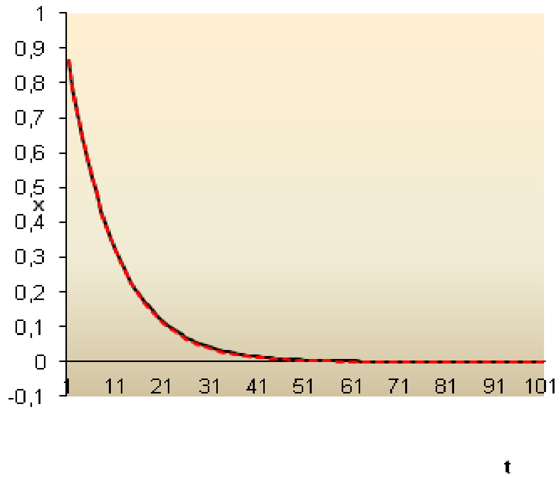

Fig. 5(A) Present perturbation solution (31) (dotted line) similar numerical solution (solid line) they are drawn with initial situations $x(0) = 1.000$, or

$\dot{x}(0) = 0.000\ \ddot{x}(0) = -0.1000, x_1 = 0.866172, y_1 = -0.817847, z_1 = -0.442186$ for $\varepsilon = 1.0$

$\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 sin2\tau)},$

$\lambda_1 = -.100,\ \lambda_2 = -7.00. \lambda_3 = -0.700$

Fig. 5(B)

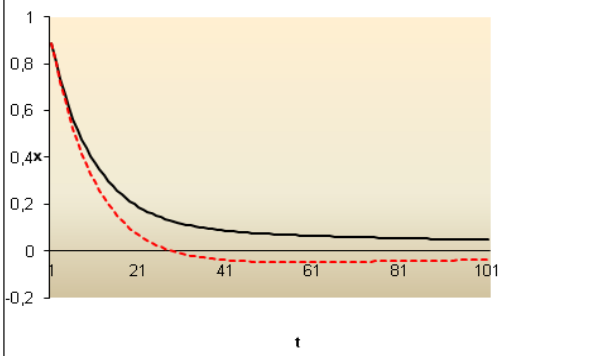

Fig. 5(B) Unified solutions (dotted line) with similar numerical solutions (solid line) are plotted with initial situations $x(0) = 1.000$, or

$\dot{x}(0) = 0.000\ \ddot{x}(0) = -0.1000\ x_1 = 0.885773,\ y_1 = -0.88836, z_1 = -1.523429$ for $= 1.0$

$\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2 \cos\tau + \zeta_3^2 sin2\tau)},$

$\lambda_1 = -.100,\ \lambda_2 = -7.00. \lambda_3 = -0.700$

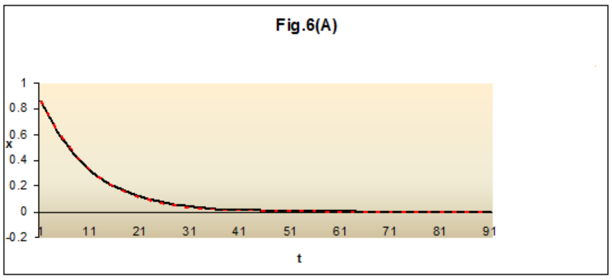

Fig. 6(A) Present perturbation solution (31) (dotted line) similar numerical solution (solid line) they are drawn with initial situations $x(0) = 1.000$, or

$\dot{x}(0) = 0.000\ \ddot{x}(0) = -0.1000 x_1 = 0.862789, y_1 = -0.806632, z_1 = -0.391573$ for $\varepsilon = 1.1$,

$$\omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2\cos\tau + \zeta_3^2 sin2\tau)},$$

$$\lambda_1 = -.100,\ \lambda_2 = -7.00.\ \lambda_3 = -0.700$$

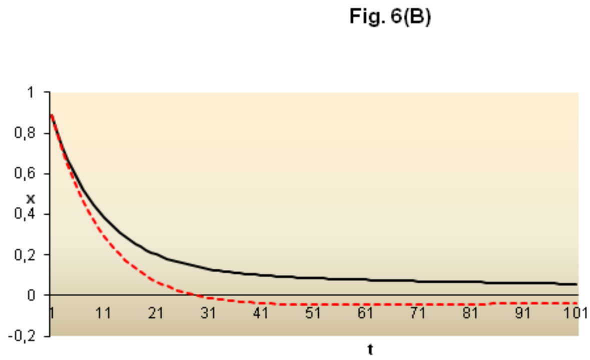

Fig. 6(B) Unified solutions (dotted line) with similar numerical solutions (solid line) are plotted with initial situations $x(0) = 1.000$, or

$\dot{x}(0) = 0.000\ddot{x}(0) = -0.1000 x_1 = 0.884351, y_1 = -0.877895, z_1 = -0.1.575339$ for

$$\varepsilon = 1.1\ \omega = \omega_0\sqrt{(\zeta_1^2 + \zeta_2\cos\tau + \zeta_3^2 sin2\tau)},$$

$$\lambda_1 = -.100,\ \lambda_2 = -7.00.\ \lambda_3 = -0.700$$

From Tables 1, 2, 3, and 4, the percent error between (15) and (23) is less than 1% for strong non-linearity and one eigenvalue is a multiple (more than 200 times) of the other eigenvalues, while the percentage errors of unified solutions [29] (A.7) and the percentage errors of unified solutions [16] (B.10) are much more than 1% and strong non-linearity causes serious problems. From Figs. 1(A), 2(A), 3(A),.4(A), 5(A) and 6(A) Perturbation solutions agree well with numerical solutions but in these situations Figs. 1(B), 1(C), 2(B), 2(C), 3(B), 3(C), 4(B), 4(C), 5(B) and 6(B) disagree, and the solution does not produce the desired result.

4. Conclusion

In conclusion, we suggest that, in this article, the extended KBM method and the HB methods have been modified and applied successfully to the second and third order autonomous nonlinear vibration systems with slowly changing coefficients. Normally, in the unified KBM method, it is noticed that much error occurs in the case of rapid changes with x respect to time t. However, all aforementioned results obtained in this paper correspond accurately to the numerical solutions obtained from the fourth order Runge-Kutta method. It is, therefore, concluded that the extended KBM method and the HB methods provide highly accurate results, which can be applied for different types of nonlinear differential systems. This article aims to establish a slowly time-varying solution of an over damped nonlinear vibration system where one eigenvalue is an integer multiple (greater than two hundred times) of the other eigenvalues. The integrated multiple eigenvalue can provide a better result than other eigenvalues for strong linearity (even if $\varepsilon \geq 1$). These methods will keep a significant contribution to future research on nonlinear vibrating problems, which emerge in mathematical physics and engineering.

Appendix A

Discussion of [29] unified theory**:**

Author's choose an approximate solution of (1) in the form

$$x(t,\varepsilon) = a(t)e^{-\lambda t} + b(t)e^{-\mu t} + \varepsilon u_1(a,b,t) + \varepsilon^2 \dots \quad \text{(A.1)}$$

where a and b satisfy the equation

$$\dot{a}_1 = \epsilon A_1(a,b,t) + \varepsilon^2 \dots$$
$$\dot{a}_2 = \epsilon B_1(a,b,t) + \varepsilon^2 \dots \quad \text{(A.2)}$$

The equations

$$(\partial/\partial t - \lambda + \mu)A_1 e^{-\lambda t} + (\partial/\partial t + \lambda - \mu)B_1 e^{-\mu t} = -(3ab^2 e^{(\lambda+2\mu)t} + b^3 e^{-3\mu t}) \quad \text{(A.3)}$$

When $\lambda \approx 3\mu$ (A3) separated into two the following equations

$$(\partial/\partial t - \lambda + \mu)A_1 e^{-\lambda t} = -b^3 e^{-3\mu t}$$
$$(\partial/\partial t + \lambda - \mu)B_1 e^{-\mu t} = -3ab^2 e^{(\lambda+2\mu)t} \quad \text{(A.4)}$$

Thus, B_1 does not contain the term $\mu \rangle 0$. However, the above functions of A_1 and B_1 are valid if μ is small. The values of A_1 and B_1 from (A.6) and then integrating with respect to t, we obtain

$$a = a_0 + b_0 \Big/ \left(1 + \varepsilon b_0^2 (e^{-2\mu t} - 1) \Big/ (3\mu^2 - \lambda\mu)\right)^{\frac{1}{2}}$$

$$b = b_0 \Big/ \left(1 + \varepsilon b_0^2 (e^{-2\mu t} - 1) \Big/ (3\mu^2 - \lambda\mu)\right)^{\frac{1}{2}} \quad \text{(A.5)}$$

$$(\partial/\partial t + \mu - \lambda)(\partial/\partial t - \mu + \lambda)u_1 = -(3a^2 b e^{(2\lambda+\mu)t} + a^3 e^{-3\lambda t}) \quad \text{(A.6)}$$

Therefore, the first order solution of (A. 1) is

$$x(t,\varepsilon) = a(t)e^{-\lambda t} + b(t)e^{-\mu t} + \varepsilon u_1 + \cdots \quad \text{(A.7)}$$

where a and b are given by (A.5) and u_1 is given by (A.6).

Appendix B

The following is a discussion of Unified Theory

[16]. The article [16] found a unified solution in the form

$$x(t,\varepsilon) = \rho cosh\psi + \varepsilon u_1(\rho,\psi) + \cdots \quad \text{(B.1)}$$

or

$$x(t,\varepsilon) = \rho sinh\psi + \varepsilon u_1(\rho,\psi) + \cdots \quad \text{(B.2)}$$

$$\dot{\rho} = -k\rho + \varepsilon A_1(\rho) + \cdots$$

$$\dot{\psi} = -\omega_0 + \varepsilon AB_1(\rho) + \cdots \quad \text{(B.3)}$$

It is notable that such unified solutions can be derived from (4). We rewrite (4) as

$$x(t,\varepsilon) = a(t)e^{-\lambda t} + b(t)e^{-\mu t} + \varepsilon u_1 + \cdots. \quad \text{(B.4)}$$

where a and b satisfy the first order differential equations

$$\left.\begin{aligned}\dot{a} &= \varepsilon \widetilde{A_1}(a,b,t) + \\ \dot{b} &= \varepsilon \widetilde{B_1}(a,b,t) + \end{aligned}\right\} \quad \text{(B.5)}$$

The roots of the linear equation are $\lambda_1 = -k + i\omega_0$ and $\lambda_2 = -k - i\omega_0$, according to the unified theory, so that

$$f^{(0)} = -e^{-3kt}(a^3 e^{3\omega_0 t} + 3a^2 b e^{\omega_0 t} + 3ab^2 e^{-\omega_0 t} + b^3 e^{-3\omega_0 t}).$$

Furthermore, with respect to the KBM method, u_1 does not contain terms with $e^{\omega_0 t}$ and $e^{-\omega_0 t}$. Replacing the values of λ_1, λ_2 and $f^{(0)}$ into (B.9) and imposing that u_1 omits the terms with $e^{\omega_0 t}$ and $e^{-\omega_0 t}$, we obtain

$$(\partial \widetilde{A_1}/\partial t + 2i\omega_0 \widetilde{A_1} = -3a^2 b e^{-2kt}$$

$$(\partial \widetilde{B_1}/\partial t + 2i\omega_0 \widetilde{B_1} = -3ab^2 e^{-2kt} \quad \text{(B.6)}$$

and

$$(\partial/\partial t + k - i\omega_0)(\partial/\partial t + k + i\omega_0)u_1 = -e^{-3kt}\left(a^3 e^{3i\omega_0 t} + b^3 e^{3i\omega_0 t}\right) \quad \text{(B.7)}$$

Replacing the values of $\tilde{A}_1$ and $\tilde{B}_1$ from (B.13) into (B.5), we obtain

$$\dot{a} = 3\varepsilon a^2 b e^{-2kt}/2(K - i\omega_0)$$

$$\dot{b} = 3\varepsilon ab^2 e^{-2kt}/2(K + i\omega_0) \quad \text{(B.8)}$$

Equations of (B.15) have exact solutions. These equations correspond to

$$\dot{r} = 3\varepsilon k r^3 b e^{-2kt}/8\omega^2$$

$$\dot{\varphi} = 3\varepsilon\omega_0 r^2 e^{-2kt}/8\omega^2 \quad \text{(B.9)}$$

Under the transformations, $a = \frac{1}{2} r e^{i\varphi}$ $b = \frac{1}{2} r e^{-i\varphi}$

However, under the above transformations (B.4) becomes

$$x(t,\varepsilon) = re^{-kt}\cos(\omega_0 + \varphi) + \varepsilon u_1 \quad \text{(B.10)}$$

where u_1 is given by (B.7) , r and φ are given by (B.9). Replace $\rho = re^{-kt}$ and $\psi = \omega_0 t + \varphi$

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