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The Effect of Combining the Adomian Decomposition and Homotopy Perturbation Methods on Solving Fifth-Order Boundary Value Problems

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Abstract: This paper considers fifth-order boundary value problems with two-point boundary conditions. For the investigation, first, the Adomian decomposition method (ADM) was applied to the equation and then the homotopy perturbation method (HPM) was used to continue the solution. In addition, this combined method was applied to solve two linear and nonlinear experiments. Then the numerical results obtained by other methods were compared. Furthermore, the convergence of the methods was analyzed. In most of the articles, the HPM is used for equations with initial conditions and the ADM is used for equations with initial and boundary conditions. In this study, a combination of two methods (the ADM and HPM) was used to solve equations with boundary conditions. The results of this combined method are remarkable.

Keywords: homotopy perturbation method, Adomian decomposition method, combination, convergence.

结合阿多米安分解和同伦摄动方法求解五阶边值问题的效果

摘要: 本文考虑具有两点边界条件的五阶边值问题。 在研究中，首先将阿多米安分解法(ADM)应用于方程，然后使用同伦摄动法(高压氧)继续求解。此外，这种组合方法被应用于求解两个线性和非线性实验。然后比较了其他方法得到的数值结果。此外，分析了方法的收敛性。在大多数文章中，高压氧用于具有初始条件的方程，而 ADM 用于具有初始条件和边界条件的方程。 在这里，结合使用两种方法（ADM 和高压氧）来求解具有边界条件的方程。这种组合方法的结果是显着的。

关键词: 同伦摄动法、阿多米安分解法、组合、收敛。

1. Introduction

A perturbation method is widely used in the analysis of nonlinear engineering problems [11], [12]. The major drawback of the perturbation method is its dependence on a small parameter. To solve this problem, other techniques such as the homotopy analysis method [9], [10], the Adomian decomposition method (ADM) [5], [8], [14] and the variation iteration method [4] were studied by researchers. The Adomian decomposition method was used for problems with boundary and initial values [5], but the homotopy perturbation method (HPM) is more effective for problems with initial conditions and it is less efficient for problems with boundary values. To solve such problems, researchers usually use other methods or indirect techniques [6], [7].

In this study, we consider fifth-order boundary value problems of the type:

$$y^{(5)}(x) = f\left(x, y, y', y'', y''', y^{(4)}\right) \quad (1)$$

with two-point boundary conditions

$$y(a) = A_1, \quad y'(a) = A_2, \quad y''(a) = A_3, \quad y(b) = B_1, \quad y'(b) = B_2$$

where f is a continuous function on [a, b] and the parameters $A_i,\ i = 1,2,3$ and $B_j,\ j = 1,2$ are real constants.

These types of problems with boundary conditions

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exist in different formats in mathematics, physics and engineering [1]-[3]. Many methods such as Galerkin, collocation [2], [9], different types of decomposition [14] and sixth order B-spline have been developed for solving fifth-order boundary value problems of the type (1) [1]. The quantic polynomial spline functions were used to solve the equation with fifth-order boundary values [3]. An analytical answer to problem (1) was obtained by the Adomian analysis method [14]. A series of answers to the fifth-order boundary value problems were obtained by the iteration method [6]. The finite difference method for these types of problems was examined in [15].

In the continuation of the study of problems with boundary values, we intended to combine the Adomian decomposition method, which is well compatible with these problems, with the homotopy perturbation method, and find a new relationship for a better solution.

The rest of the manuscript is presented as follows. Section 2 analyzes the ADM and HPM methods. The convergence of the current study is given in Section 3. Section 4 examines the methods with numerical experiments. Finally, a brief conclusion is depicted in Section 5.

2. Method Analysis (ADM-HPM)

To illustrate the idea for solving problems with boundary values, we consider the following combination of linear and nonlinear differential equations:

$$\mathrm{Lu} + \mathrm{Ru} + \mathrm{Nu} = \mathrm{g}, \quad \mathrm{x} \in \Omega \tag{2}$$

with the initial and boundary conditions

$$\mathrm{B}\left(\mathrm{u}, \frac{\partial \mathrm{u}}{\partial \mathrm{n}}\right) = 0\,, \quad \mathrm{x} \in \Gamma \tag{3}$$

where L is the linear operator with highest degree n, R is the linear operator with less than L degree, N is the nonlinear operator, and g is a known function. Γ is the boundary of the region Ω *of problem (2).* Assume that L is a derivative of order n, where $L = \frac{d^n}{dx^n}$, and by applying L^{-1} to equation (2) we have:

$$L^{-1}\mathrm{Lu} = L^{-1}\big(g(x)\big) - L^{-1}Ru - L^{-1}Nu, \tag{4}$$

from the expansion of relation (4) we have:

$$u = a_{0,n-1}\frac{x^{n-1}}{(n-1)!} + a_{0,n-2}\frac{x^{n-2}}{(n-2)!} + \cdots + a_{0,1}x + a_{0,0} + L^{-1}\big(g(x)\big) - L^{-1}Ru - L^{-1}Nu \tag{5}$$

From equation (5) we have:

$$u_0 = a_{0,n-1}\frac{x^{n-1}}{(n-1)!} + a_{0,n-2}\frac{x^{n-2}}{(n-2)!} + \cdots + a_{0,1}x + a_{0,0} + L^{-1}\big(g(x)\big), \tag{6}$$

Now, from the application of boundary conditions (3) to relation (6), we obtain the unknown values of $a_{0,0},\ a_{0,1}, \dots,\ a_{0,n-2},\ a_{0,n-1}$ and replace them in u_0.

Assuming

$$v = \sum_{i=0}^{+\infty} p^i u_i. \tag{7}$$

For the polynomial relation, we used [13] for its nonlinear part as:

$$N\Big(\sum_{i=0}^{+\infty} p^i u_i\Big) = \sum_{i=0}^{\infty} A_i(u_0, u_1, \cdots, u_i)\, p^i,$$

which polynomials A_n were defined by

$$A_n(u_0, u_1, \cdots, u_n) = \Big[\frac{1}{n!}\frac{d^n}{\lambda^n} N\Big(\sum_{k=0}^{+\infty} \lambda^k u_k\Big)\Big]_{\lambda=0}\,, \quad n = 0,1,2,\cdots.$$

To continue solving the problem, we use the HPM for equation (5), and obtain:

$$H(v,p) = v - u_0 + p[L^{-1}Rv + L^{-1}Nv]. \tag{8}$$

Substituting (7) into (8) and rewriting Eq. (8), we have:

$$\sum_{i=0}^{+\infty} p^i u_i = u_0 - p\Big[L^{-1}R\Big(\sum_{i=0}^{+\infty} p^i u_i\Big) + L^{-1}\Big(\sum_{i=0}^{\infty} A_i(u_0, u_1, \cdots, u_i)\, p^i\Big)\Big], \tag{9}$$

By equating the coefficients p to zero in relation (9), we have:

$p^0 : w_0 = u_0$

$$p^1 :\ u_1 = a_{1,n-1}\frac{x^{n-1}}{(n-1)!} + a_{1,n-2}\frac{x^{n-2}}{(n-2)!} + \cdots + a_{1,1}x + a_{1,0} + L^{-1}Ru_0 + L^{-1}A_0(u_0)$$

Assuming

$$w_1 = u_0 + u_1, \tag{10}$$

By applying the boundary conditions (3) to relation (10), we calculate the unknown values $a_{1,0},\ a_{1,1}, \dots,\ a_{1,n-2},\ a_{1,n-1}$ and replace in u_1.

Hence, we have the following:

$$p^2 :\ u_2 = a_{2,n-1}\frac{x^{n-1}}{(n-1)!} + a_{2,n-2}\frac{x^{n-2}}{(n-2)!} + \cdots + a_{2,1}x + a_{2,0} + L^{-1}Ru_1 + L^{-1}A_1(u_0, u_1).$$

Assuming $w_2 = u_0 + u_1 + u_2$ and applying the boundary conditions on it, we obtain unknown values and replace them in u_2. In the continuation of this process, by obtaining the values of $u_3, u_4, \cdots$, the answer to the problem is obtained as follows:

$$u = \lim_{p \to 1} v = \sum_{i=0}^{+\infty} u_i.$$

3. Convergence of the Current Study

To study the convergence of the method, we considered the following theorems [16].

Theorem 1: (Sufficient Condition of Convergence) Suppose that X and Y are Banach spaces and $N: X \to Y$ is a contractive nonlinear mapping, that is

$$\forall\, w, v \in X; \quad \|N(w) - N(v)\| \le \gamma \|w - v\|\,, \quad 0 < \gamma < 1.$$

Then, according to Banach's fixed point theorem, N has a unique fixed point u, that is N(u) = u.

If the sequence produced by the AMD-HPM method is as follows:

$$s_n = N(s_{n-1}),\ \ s_{n-1} = \sum_{i=0}^{n-1} u_i, \quad n = 1, 2, 3, \dots,$$

And suppose $s_0 = u_0 \in B_r(u)$ *that* $B_r(u) =$

$\{u^* \in X \mid \|u^* - u\| < r\}$, then we have:

a) $s_n \in B_r(u)$,

b) $\lim_{n\to\infty} s_n = u$.

Proof: Part (a) is easily proved by induction. For part (b) we have:

$$\|s_n - u\| = \|N(s_n) - N(u)\| \le \gamma\|s_{n-1} - u\| \le \cdots \le \gamma^n\|s_0 - u\|,$$

because $\lim_{n\to\infty} \gamma^n = 0$ then $\lim_{n\to\infty} s_n = u$.

Theorem 2: Let X be a Banach space, $\sum_{i=0}^{\infty} u_i$ obtained by (9), convergence to $u \in X$, if

$$\exists(0 \le \gamma < 1), \mid \forall n \in N \Longrightarrow \|u_n\| \le \gamma\|u_{n-1}\|.$$

where *u* is the exact solution.

Proof: We have

$$\|s_{n+1} - s_n\| = \|u_{n+1}\| \le \gamma\|u_n\| \le \cdots \le \gamma^n\|u_0\|.$$

Therefore,

$$\lim_{n,m\to\infty}\|s_n - s_m\| = 0.$$

Then, $\|s_n\|$, is the Cauchy sequence in the Banach space and $\exists$ u $\in$ X s.t $\lim_{n\to\infty} s_n = u$.

4. Numerical Experiments

In this section, we consider two linear and nonlinear experiments for the application of the above method and compare the results at the end of each experiment.

Experiment 1 [14]: Consider the fifth-order linear differential equation:

$$y^{(5)}(x) = y - 15e^x - 10xe^x,\ 0 < x < 1, \qquad (11)$$

With the boundary conditions

$$y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,$$
$$y(1) = 0, \quad y'(1) = -e, \qquad (12)$$

the analytic solution is $y = x(1 - x)e^x$.

By converting Equation (11) to the following form:

$$Ly = y - 15e^x - 10xe^x, \qquad (13)$$

where $L = \frac{d^5}{dx^5}$ and its inverse effects on relation (13), we have:

$$y(x) = a_{0,4}\frac{x^4}{24} + a_{0,3}\frac{x^3}{6} + a_{0,2}\frac{x^2}{2} + a_{0,1}x + a_{0,0} - L^{-1}y - L^{-1}(15e^x + 10xe^x), \qquad (14)$$

From equation (14), we have:

$$y_0 = a_{0,4}\frac{x^4}{24} + a_{0,3}\frac{x^3}{6} + a_{0,2}\frac{x^2}{2} + a_{0,1}x + a_{0,0} - L^{-1}(15e^x + 10xe^x) \qquad (15)$$

By applying the boundary conditions (12) to relation (15) and obtaining the unknown values $a_{0,0}$, $a_{0,1}$, $a_{0,2}$, $a_{0,3}$, $a_{0,4}$, we have:

$$y_0 = -.1213721518x^4 - 1.335673548x^3 - 7.5x^2 - 24x - 35 - 5(2x - 7)e^x$$

To continue solving the problem, we apply the HPM to equation (14), i.e.,

$$H(v, p) = v - y_0 - p(L^{-1}v), \qquad (16)$$

where $v = \sum_{i=0}^{+\infty} p^i y_i$. If we set the coefficient p in equation (16) equal to zero, we have:

$$p^0 : w_0 = y_0,$$

$$p^1 :\ y_1 = a_{1,4}\frac{x^4}{24} + a_{1,3}\frac{x^3}{6} + a_{1,2}\frac{x^2}{2} + a_{1,1}x + a_{1,0} + L^{-1}y_0$$

Assuming that:

$$w_1 = y_0 + y_1 = a_{1,4}\frac{x^4}{24} + a_{1,3}\frac{x^3}{6} + a_{1,2}\frac{x^2}{2} + a_{1,1}x + a_{1,0} - 35 + 120e^x - 24x - 20xe^x - 7.5x^2 - 1.3356735x^3 - .12137215x^4 - .29166666x^5 - 0.03333333x^6 - 0.00297619x^7 - 0.00019876x^8 - 0.00000802x^9 \qquad (17)$$

By applying boundary conditions (12) to relation (17) and obtaining the unknown values $a_{1,0}$, $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{1,4}$ and replacing them in y_1, we have:

$$y_1 = -85 + 85e^x - 75x - 10xe^x - 32.5x^2 - 9.1643265x^3 - 1.8786275x^4 - .29166666x^5 - 0.03333333x^6 - 0.00297619x^7 - 0.00019876x^8 - 0.00000802x^9,$$

In the following, we have:

$$p^2 :\ y_2 = a_{2,4}\frac{x^4}{24} + a_{2,3}\frac{x^3}{6} + a_{2,2}\frac{x^2}{2} + a_{2,1}x + a_{2,0} + L^{-1}y_1$$

Assuming that:

$$w_2 = y_0 + y_1 + y_2 = y_0 + y_1 + a_{2,4}\frac{x^4}{24} + a_{2,3}\frac{x^3}{6} + a_{2,2}\frac{x^2}{2} + a_{2,1}x + a_{2,0} + L^{-1}y_1 \qquad (18)$$

By obtaining the unknowns of relation (18), the solution y_2 is obtained.

In the continuation of this process, after a few steps, the solution to problem (11) is obtained as follows:

$$y = y_0 + y_1 + y_2 + \cdots = 440e^x - 40xe^x - 440 - 399x - 180x^2 - 53.833331x^3 - 12.000001x^4 - 2.12500x^5 - .31111111x^6 - 0.03869047x^7 - 0.00416666x^8 - 0.00039406x^9 - 0.00003306x^{10} - 0.00000241x^{11} - 0.0000001x^{12} - 0.00000001x^{13} + O(x^{14}).$$

In Table 1, the solutions of Experiment 1 with the proposed method (ADM-HPM) are compared to those obtained by the B-spline method [1].

Table 1 Experiment 1 error estimates

x_i	y(exact)	Errors (ADM-HPM)	Errors (B-spline)
0.0	0.0000000000	0.000000	0.000
0.1	0.0994653826	5.109000e-8	8.0e-3
0.2	0.1954244413	7.530000e-8	1.2e-3
0.3	0.2834703497	2.003000e-7	5.0e-3
0.4	0.3580379275	2.224000e-7	3.0e-3
0.5	0.4121803178	2.481000e-7	8.0e-3
0.6	0.4373085120	8.560000e-8	6.0e-3
0.7	0.4228880685	1.607000e-7	0.000
0.8	0.3560865485	2.058000e-7	9.0e-3
0.9	0.2213642800	6.016000e-7	9.0e-3
1.0	0.0000000000	6.982649e-8	0.000

Suppose that

$$w_n = N(w_{n-1}), \qquad w_{n-1} = \sum_{i=0}^{n-1} y_i\,, \quad n = 1,2,3,\dots,$$
$$w_0 = y_0,$$
$$w_n = \sum_{i=1}^{n} [a_{i,4}\frac{x^4}{24} + a_{i,3}\frac{x^3}{6} + a_{i,2}\frac{x^2}{2} + a_{i,1}x + a_{i,0} + \int\int\int\int\int (y_{i-1})\, dxdxdxdxdx], \quad n = 1,2,\cdots.$$

According to Theorem 1 for the nonlinear mapping N, considering $\|g(x)\| = \max_{0<x<1} |g(x)|$, we have:

$\|w_0 - y\| =$
$\|-.1213721518x^4 - 1.335673548x^3 - 7.5x^2 - 24x - 35 - 52x - 7ex - x(1-x)ex,$

and $\|w_1 - y\| \le \|w_0 - y\| \left\|\frac{w_1-y}{w_0-y}\right\|, \forall\, x \in (0,1)$, $\left\|\frac{w_1-y}{w_0-y}\right\| \le 0.9 = \gamma < 1$, thus,

$\|w_1 - y\| \le \gamma \|w_0 - y\|$,

and $\|w_2 - y\| \le \|w_1 - y\| \left\|\frac{w_2-y}{w_1-y}\right\|$, $\forall\, x \in (0,1)$, $\left\|\frac{w_2-y}{w_1-y}\right\| \le 0.5 < \gamma$, thus,

$\|w_2 - y\| \le \gamma^2 \|w_0 - y\|$,

⋮

$\|w_n - y\| \le \gamma^n \|w_0 - y\|$.

Therefore,

$\lim_{n\to\infty} \|w_n - y\| \le \lim_{n\to\infty} \gamma^n \|w_0 - y\| = 0.$

Experiment 2 [14]: Consider the following nonlinear fifth-order boundary value problem

$$y^{(5)}(x) = e^{-x} y^2, \qquad 0 < x < 1 \tag{19}$$

with the boundary conditions

$$y(0) = 1,\ y'(0) = 1,\ y''(0) = 1,\ y(1) = e,\ y'(1) = e, \tag{20}$$

The analytic solution is $y = e^x$.

Using the ADM on relation (19):

$$Ly = e^{-x} y^2, \tag{21}$$

where $L = \frac{d^5}{dx^5}$ and its inverse effects on relation (21) we have:

$$y(x) = a_{0,4}\frac{x^4}{24} + a_{0,3}\frac{x^3}{6} + a_{0,2}\frac{x^2}{2} + a_{0,1}x + a_{0,0} + L^{-1}e^{-x}y^2 \tag{22}$$

From relation (22) we have:

$$y_0 = a_{0,4}\frac{x^4}{24} + a_{0,3}\frac{x^3}{6} + a_{0,2}\frac{x^2}{2} + a_{0,1}x + a_{0,0}, \tag{23}$$

By applying the boundary conditions (20) to relation (23) and obtaining the unknown values $a_{0,0},\ a_{0,1},\ a_{0,2},\ a_{0,3},\ a_{0,4}$, we have:

$$y_0 = 1 + x + 0.5x^2 + 0.1548454841x^3 + 0.0634363439x^4.$$

For the nonlinear part of equation (19), we use He polynomials to linearize the problem, we have:

$A_0(y_0) = y_0^2,$
$A_1(y_0, y_1) = 2y_0 y_1,$
$A_2(y_0, y_1, y_2) = y_1^2 + 2y_0 y_2,$
⋮

To apply the HPM, we write the nonlinear part of the problem as follows:

$$y^2 = Ny = \sum_{i=0}^{\infty} p^i A_i(y_0, y_1, \cdots, y_i) = y_0^2 + 2py_0y_1 + p^2(y_1^2 + 2y_0y_2) + \cdots. \tag{24}$$

To continue solving the problem, we use the HPM on relation (22) and using relation (24) and $v = \sum_{i=0}^{+\infty} p^i y_i$.

$$H(v,p) = v - y_0 - pL^{-1}(e^{-x}Ny). \tag{25}$$

If we set the coefficient p in Eq. (25) equal to zero, we have:

$p^0 : w_0 = y_0,$

$$p^1:\ y_1 = a_{1,4}\frac{x^4}{24} + a_{1,3}\frac{x^3}{6} + a_{1,2}\frac{x^2}{2} + a_{1,1}x + a_{1,0} + L^{-1}(e^{-x}y_0^2),$$

Assuming that:

$$w_1 = y_0 + y_1 = a_{1,4}\frac{x^4}{24} + a_{1,3}\frac{x^3}{6} + a_{1,2}\frac{x^2}{2} + a_{1,1}x + a_{1,0} + 1 + x + .5x^2 + .1548454841x^3 + 0.0634363439x^4 + e^{-x}(-0.0040241697x^8 - .1806124519x^7 - 4.155314237x^6 - 62.60512573x^5 - 664.7035600x^4 - 5029.202944x^3 - 26214.27018x^2 - 85350.13030x - 131966.6125), \tag{26}$$

By applying boundary conditions (20) of relation (26) and obtaining the unknown values $a_{1,0}$, $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{1,4}$ and replacing them in y_1, we have:

$$y_1 = 131966.61 - 46616.482x + 6847.4461x^2 - 504.42573x^3 + 16.201119x^4 + e^{-x}(-0.0040241x^8 - .18061245x^7 - 4.1553142x^6 - 62.605125x^5 - 664.70356x^4 - 5029.2029x^3 - 26214.270x^2 - 85350.130x - 131966.61),$$

In the continuation of this process, after a few steps, the answer to problem (19) is obtained as follows:

$$y = y_0 + y_1 + \cdots = 131967.61 - 46615.482x + 6847.9461x^2 - 504.27089x^3 + 16.264556x^4 + e^{-x}(-0.004024x^8 - .18061245x^7 - 4.1553142x^6 - 62.605125x^5 - 664.70356x^4 - 5029.2029x^3 - 26214.270x^2 - 85350.130x - 131966.61) + \cdots.$$

In Table 2, the solutions of Experiment 2 obtained by the proposed method (ADM-HPM) are compared with those obtained by the B-spline method [1].

Table 2 Experiment 2 error estimates

x_i	y(exact)	Errors (ADM-HPM)	Errors (B-spline)
0.0	1.0000000000	0.000000e+00	0.0000
0.1	1.1051709180	2.908200e-05	7.0e-4
0.2	1.2214027580	2.758000e-06	7.2e-4
0.3	1.3498588080	4.119200e-05	4.1e-4
0.4	1.4918246980	7.530200e-05	4.6e-4
0.5	1.6487212710	2.127100e-05	4.7e-4
0.6	1.8221188000	1.880000e-05	4.8e-4
0.7	2.0137527070	5.270700e-05	3.9e-4
0.8	2.2255409280	4.092800e-05	3.1e-4
0.9	2.4596031110	3.111000e-06	1.6e-4
1.0	2.7182818280	1.817200e-05	0000.0

According to Theorem 2 for the nonlinear mapping N, considering $\|g(x)\| = \max_{0<x<1} |g(x)|$, we have:

$$\left\|\frac{y_1}{y_0}\right\| \leq 0.8 = \gamma < 1 \Rightarrow \|y_1\| \leq \gamma\|y_0\|,$$

$$\left\|\frac{y_2}{y_1}\right\| \leq 0.02 < \gamma \Rightarrow \|y_2\| \leq \gamma^2\|y_0\|,$$

$$\vdots$$

$$\left\|\frac{y_n}{y_{n-1}}\right\| \leq \gamma \Rightarrow \|y_n\| \leq \gamma^n\|y_{n-1}\|.$$

Therefore,

$$\lim_{n\to\infty} \|y_n\| \leq \lim_{n\to\infty} \gamma^n \|y_0\| = 0.$$

5. Conclusions

Differential problems with boundary values are widely used in basic sciences and engineering. Decomposition methods, such as Adomian decomposition, variational iteration, successive approximations, and iteration methods, such as Galerkin, collocation and sixth order B-spline, are used to solve problems with boundary values. Each of these methods has its strengths and weaknesses.

The HPM is more effective for problems with initial conditions. But the ADM is also used for boundary values. Therefore, we solved the problems with boundary values by combining the HPM and ADM. In this study, the problems with fifth-order boundary values were investigated.

The good results in solving the above experiments show that this idea can be used to solve problems with boundary values in general.

A weakness of these methods is their relationship with the Fourier series, which is more accurate and stable for points close to zero. It is necessary to have computers with high processing power to solve the algorithms of these methods and equations.

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