1. Introduction

Laplace transfigures of a complex or real-valued function y (t) of one variable t, t> 0 indicated as L (y (t)) is defined as:

$$L(y(t)) = y(s) = \int_0^{\infty} e^{-st} y(t)dt \qquad (1)$$

We can see from the resources mentioned in this manuscript that lower interest has been planted in the analysis of numerical inversion of Laplace transform using the functional matrix. However, abundant functional matrices of orthogonal and non-orthogonal polynomials are applied to linear and nonlinear differential and integral equations like Chebyshev, Legender, Jacobi, Bernstein, Bernoulli, Genocchi, Lucas, Laguerre, Hermite, Bell, etc. Haar Wavelet [1] developed inverse Laplace transform numerically through some of the functional matrices. Haar Wavelet was modified using generalized block palpitation function, and functional matrix supported Piecewise constant block palpitation function [3], [4].

A numerical inverse Laplace transfigure system is established in [10] using the Bernoulli polynomials functional integration matrix.

This paper aims to apply a model for the numerical inverse of the Laplace transform using the Chebyshev polynomials functional matrix.

2. Elements of Study

2.1. Chebyshev Polynomials

A sequence of orthogonal polynomials which are related to de Moivre's formula and which can be defined recursively is called the Chebyshev polynomials. In [2], the *n*-th degree of the Chebyshev polynomials is defined by:

$$T_n(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{n!}{(2m)!\,(n-2m)!} (1-x^2)^m x^{n-2m} \qquad (2)$$

where

$$[n/2] = \begin{cases} n/2 & \text{if n is even} \\ (n+1)/2 & \text{if n is odd} \end{cases}$$

$T_0(x) = 1,$
$T_1(x) = x,$
$T_2(x) = 2x^2 - 1,$
$T_3(x) = 4x^3 - 3x,$
$T_4(x) = 8x^4 - 8x^2 + 1,$
$T_5(x) = 16x^5 - 20x^3 + 5x,$
$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

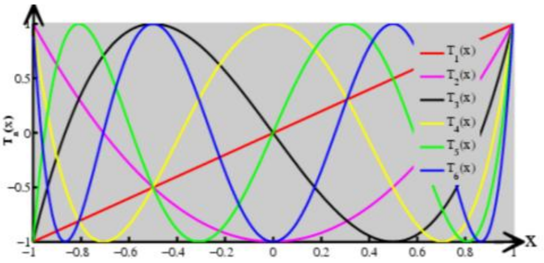

Fig. 1 First 6 Chebyshev polynomials over the interval [-1, 1]

Now, we define

$$\Phi(x) = [T_{0,n}(x), T_{1,n}(x), \ldots,\ T_{n,n}(x)]^T \qquad (3)$$

where

$$\Phi(x) = AB_n(x) \qquad (4)$$

That A is an $(n+1)X(n+1)$ upper triangular matrix with rows

$$\left[\overbrace{0,0,\ldots 0}^{i\ times}, (-1)^0\binom{n}{i}\binom{n-i}{0}, (-1)^1\binom{n}{i}\binom{n-i}{1}, \ldots, (-1)^{m-i}\binom{n}{i}\binom{n-i}{n-i}\right]$$

$B_n(x)$ is a $(n+1) \times 1$ matrix as follows:

$$B_n(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}$$

2.2. Function Approximation

Obtaining approximate function values is much quicker with an approximating functional form. Assume that the Hilbert space with the inner product is $H = L^2[0,1)$, defined by:

$$(f,g) = \int_0^1 f(x)g(x)dx$$

and

$$Y = \mathrm{Span}\{T_{0,n}(x), T_{1,n}(x), \ldots,\ T_{n,n}(x)\}$$

Here Y is a complete subspace of H if it is a finite-dimensional and closed subspace.

Now f has a unique best approximation out of Y such as y_0, if it is any arbitrary element in H, that is $\exists y_0 \in Y \;\; s.t \;\; \forall y \in Y \;\; \|f-y\|$

This gives

$$\forall y \in Y \qquad (f - y_0) = 0 \qquad (5)$$

Since $\forall y \in Y$ so, there exist coefficients $c_0, c_1, \ldots, c_n$ such that

$$y_0 = c^T\Phi(x),$$

where

$$c^T = [c_0, c_1, \ldots, c_n] \qquad (6)$$

By (5)

$$\left(f - \boldsymbol{c^T}\Phi(x), T_{i,n}(x)\right) = 0, \qquad i = 0,1,\ldots,n,$$

If we write

$$\left(\boldsymbol{c^T}(\Phi(x),\Phi(x)\right) = (f,\Phi(x)) \qquad (7)$$

where

$$(f,\Phi(x)) = \int_0^1 f(x)\Phi(x)dx$$

Dual matrix of $\Phi(x)$ is the $(n+1)\times(n+1)$ $(\Phi(x),\Phi(x))$. Assume that

$$D = (\Phi(x),\Phi(x))A\left[\int_0^1 T_n(x)T_n^T(x)dx\right]A^T = AHA^T \qquad (8)$$

H is a Hilbert matrix. We can write the element of D as:

$$D_{(i+1),\ (j+1)} = \int_0^1 T_{i,n}(x)T_{j,n}(x)dx = \frac{\binom{n}{i}\binom{n}{j}}{(2n+1)\binom{2n}{i+j}} \quad (9)$$

where $i, j = 0, 1, \dots, n$. Any function $f(x) \in L^2[0,1]$ can be written using Chebyshev basis as $f(x) \simeq c^T(\Phi(x)$, where from (7) and (8), we receive

$$c = D^{-1}(f, \Phi(x)) \quad (10)$$

The function $k(x, s) \in L^2[0,1] \times L^2[0,1]$ can be approximated as:

$$k(x, s) \simeq \Phi^T(x)K\Phi(s) \quad (11)$$

where

$$K_{i,j} = \frac{(\Phi_i(x),(k(x,s),\Phi_j(s)))}{(\Phi_i(x),\Phi_i(x))(\Phi_j(x),\Phi_j(x))} \quad (12)$$

is an $(n + 1) \times (n + 1)$ matrix for $i, j = 0, 1, \dots, n$. As follows from (8),

$$K = D^{-1}\left(\Phi_i(x), \quad (k(x, s), \Phi(s))\right) D^{-1} \quad (13)$$

2.3. Operational Matrix of Integration

The approach is grounded on reducing the differential equations into integral equations through integration, approaching colorful signals involved in the equation by abbreviated orthogonal series, and using the functional integration matrix to exclude the integral operations.

Integrating vector $\Phi(x)$ in (4), we obtain

$$\int_0^x \Phi(x')d\,x' \simeq P\,\Phi(x) \quad (14)$$

The $(n + 1) \times (n + 1)$ operational matrix for integration P in (14) is given in as:

$$\int_0^x \Phi(x')d\,x' = A_p X_p \quad (15)$$

And A_p is the $(n + 1) \times (n + 1)$ matrix,

$$A_p = A\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{n+1} \end{bmatrix} \text{and } X_p = \begin{bmatrix} x \\ x^2 \\ \vdots \\ x^{n+1} \end{bmatrix} \quad (16)$$

The elements of vector X_p in terms of $\Phi(x)$ can be approximated as: from (4), $T_n(x) = A^{-1}\Phi(x)$ then for $k = 0, 1, \dots, n$,

$$x^k = A^{-1}_{[k+1]}\Phi(x),$$

where $A^{-1}_{[k+1]}$ is $k + 1 -$ th row for A^{-1} for $k = 0, 1, \dots, n$. We just need to approximate $x^{n+1} \simeq c^T_{n+1}\Phi(x)$. Using (10) and (9), we have

$$c_{n+1} = D^{-1}\int_0^1 x^{n+1}\,\Phi(x)dx = \frac{D^{-1}}{2n+2}\begin{bmatrix} \frac{\binom{n}{0}}{\binom{2n+1}{n+1}} \\ \frac{\binom{n}{1}}{\binom{2n+1}{n+2}} \\ \vdots \\ \frac{\binom{n}{n}}{\binom{2n+1}{2n+1}} \end{bmatrix}$$

Let

$$T = \begin{bmatrix} A^{-1}_{[2]} \\ A^{-1}_{[3]} \\ \vdots \\ A^{-1}_{[n+1]} \\ c^T_{n+1} \end{bmatrix} \quad (17)$$

Then, $X_p \simeq T\Phi(x)$. Therefore, the operational matrix of integration is given by:

$$P = A_p T.$$

2.4. Product Operational Matrix

It is important to calculate the product of $\Phi(x)$ and $\Phi(x)^T$, named the product matrix of Chebyshev polynomials basis. Assume

$$\Pi(x) = \Phi(x)\Phi(x)^T \quad (18)$$

If the matrix $\Pi(x)$ is multiplying in vector c defined in (6), we have:

$$c^T\Pi(x) = \Phi(x)^T\hat{C} \quad (19)$$

where $\hat{C}$ is an $(n + 1) \times (n + 1)$ matrix and called the coefficient matrix. Thus, we have

$$c^T\Pi(x) = \left[\sum_{i=0}^n c_iT_{i,n}(x), \sum_{i=0}^n c_ixT_{i,n}(x), \dots, \sum_{i=0}^n c_ix^nT_{i,n}(x),\right]A^T \quad (20)$$

To approximate the function $x^kT_{i,n}(x)$ in terms of $\Phi(x)$, we assume that

$$e_{k,i} = \begin{bmatrix} e_0^{k,i} \\ e_1^{k,i} \\ \vdots \\ e_n^{k,i} \end{bmatrix} \quad (21)$$

By (11), we have $x^kT_{i,n}(x) \simeq e_{k,i}\Phi(x)$, $i, k = 0, 1, \dots, n$. By using (21) and (19) for $i, k = 0, 1, \dots, n$, we have

$$e_{k,i} = \int_0^1 x^kT_{i,n}(x)\Phi(x)dx\,\frac{D^{-1}\binom{n}{i}}{2n+k+1}\begin{bmatrix} \frac{\binom{n}{0}}{\binom{2n+k}{i+k}} \\ \frac{\binom{n}{1}}{\binom{2n+k}{i+k+1}} \\ \vdots \\ \frac{\binom{n}{n}}{\binom{2n+k}{i+k+n}} \end{bmatrix} \quad (22)$$

Therefore,

$$\sum_{i=0}^n c_ix^kT_{i,n}(x) \simeq \sum_{i=0}^n c_i\left(\sum_{j=0}^n e_j^{k,i}\ T_{i,n}(x)\right) = \Phi(x)^T\begin{bmatrix} \sum_{i=0}^n c_ie_0^{k,i} \\ \sum_{i=0}^n c_ie_1^{k,i} \\ \vdots \\ \sum_{i=0}^n c_ie_n^{k,i} \end{bmatrix} \quad (23)$$

$$= \Phi(x)^T[e_{k,0}, e_{k,1}, \dots, e_{k,n}\ ]c = \Phi(x)^TE_{k+1}c$$

where $E_{k+1}c$ is an $(n + 1) \times (n + 1)$ matrix, that has vectors $e_{k,i}, k = 0,1, \dots, n$ for each column. Then we define $\widetilde{E_{k+1}} = E_{k+1}c$ for k = 0,1,…,n.

If we choose an $(n + 1) \times (n + 1)$ matrix $\tilde{C} = [\widetilde{E_1}, \widetilde{E_2}, \dots, \widetilde{E_{n+1}}]$, then by (20) and (23) we have

$$c^T\Pi(x) \simeq \Phi(x)^T\tilde{C}A^T \quad (24)$$

3. Method of Solution

Now, we use the numerical inverse Laplace transform system grounded on Chebyshev polynomials. Hence, we use the functional integration matrix of Chebyshev polynomials. We introduce the process in this paper by the simplest linear system of time-varying

$$y'(t) + \mu y(t) = u(t), \qquad y(0) = 0$$

Considering $u(t)$ is the identity step function.

Now by transforming the previous differential equation into an integral equation, we got,

$$y(t) = \mu \int_0^t y(x)dx = \int_0^t u(x)dx \qquad (25)$$

Applying the Laplacian transformation on both sides of (25) and adapting the characteristics of the Laplace transform, we obtain

$$Y(s) = \frac{1}{s(s+\mu)} \qquad (26)$$

Reformulating the equation (26) in terms of $1/s$ gives

$$Y(s) = \frac{\frac{1}{s^2}}{(1+\frac{\mu}{s})} = \widetilde{Y}\left(\frac{1}{s}\right)$$

From Laplace transform theory, we have if $L(y(t)) = Y(s)$, then

$$L\left(\int_0^t y(t)dt\right) = \frac{1}{s}(y(s))$$

The consequence 1⁄s is replaced by P_(n+1), i.e., functional matrix of integration of Chebyshev polynomials and denotes the function as (Y)˜ in [3], because the integration in the time domain of the inverse Laplace transform function is corresponding to the description of the function (Y)˜ defined onto the space of matrices. Therefore, $\widetilde{Y}(P_{n+1}) = {P_{n+1}}^2(I+\mu P_{n+1})^{-1}$

By approximating the function

$$y_n(t) = F^T T(t)$$

We can solve (25) and

$$\int_0^t y_n(x)dx = F^T P_{n+1} T(t)$$

Also

$$\int_0^t u(x)dx = G^T P_{n+1} T(t)$$

where $G = [g_0 \quad g_1 \quad g_2 \dots g_n]^T$ is defined by

$$g_i = \int_0^t \beta_1(t)dt, \quad i = 0,1,\dots,n, \quad 0 \le t \le 1.$$

The given integral equation (25) takes the form:

$$F^T\ T(t) + \mu F^T P_{n+1} T(t) = G^T P_{n+1} T(t)$$

$$F^T(I + \mu P_{n+1}) T(t) = G^T P_{n+1} T(t)$$

$$F^T = G^T P_{n+1}(I+\mu P_{n+1})^{-1}$$

$$= G^T P_{n+1} P_{n+1}^{-2}[P_{n+1}^2(I+\mu P_{n+1})^{-1}]$$

$$F^T = G^T P_{n+1}^{-1} \widetilde{Y}(P_{n+1})$$

The unknown coefficient vector F It is calculated first; hence, the approximate solution $y_n(t)$ in terms of Chebyshev polynomials by substituting this vector.

3.1. Error Analysis

Theorem: Consider the function $y:[0,L]$ and $y \in C^{n+1}[0,L]$. Let y_n be the best approximate result of y, then

$$\| y - y_n \|_2 \le \frac{M}{(n+1)!}\sqrt{\frac{L^{3+2n}}{3+2n}}$$

For the proof, check [10].

4. Numerical Examples

In this section, we will solve the nonlinear differential equations, particularly oscillator equations (Duffing equation and Van der Pol equation), Blasius equation, and jerk equation by taking n = 7.

4.1. Applications to Oscillator Equations

The physics and engineering, nonlinear differential equations play a crucial role in applied mathematics. Inclusive methods are developed yet within the literature to unravel these equations. The Laplace decomposition method to find numerical solution of Duffing equations was used. The homotopy analysis approach and homotopy Padé technique for solving Duffing equations have been utilized. The uniform Haar wavelet approximation and quasilinearization procedure were introduced in [1] for solving several nonlinear oscillator equations.

In [5], the polynomial least square process has been utilized for nonlinear oscillator equations. Moreover, various other procedures like the harmonic balance method [6], improved collocation method in terms of shifted Chebyshev polynomials [7], and a block multi-step method with variable order step [8] were also applied to solve these equations. This section introduces a solution to some oscillator equations using our developed technique [10].

Example 1: The unforced Van der Pol oscillator equation is given by

$$y''(t) + k(1 - y^2(t))y'(t) + \beta^2 y(t) = 0$$

with initial conditions $y(0) = a$, $y'(0) = b$, where $\kappa, \beta, a,$ and b are defined constant values.

Table 1 presents the numerical solution obtained by our method besides Runge–Kutta method, Gear method, Chebyshev series, and Haar wavelet by choosing the parameters $k = -0.05, \beta = 1, a = 0,$ and $b = 05$.

Table 1 Numerical results of Example 1

t	Runge–Kutta method	Gear method	Haar wavelet	Presented method
0	0	0	0	2.22371E−07
0.1	0.05011	0.05012	0.05012	0.050042733

Continuation of Table 1

0.2	0.09983	0.09983	0.09983	0.099832781
0.3	0.14886	0.14886	0.14886	0.148873899
0.4	0.19665	0.19665	0.19665	0.196669120
0.5	0.24270	0.24270	0.24270	0.242734234
0.6	0.28653	0.28653	0.28653	0.286598547
0.7	0.32770	0.32770	0.32770	0.327812816
0.8	0.36577	0.36577	0.36577	0.365949431
0.9	0.40034	0.40034	0.40034	0.400611569
1	0.43105	0.43105	0.43105	0.431434743

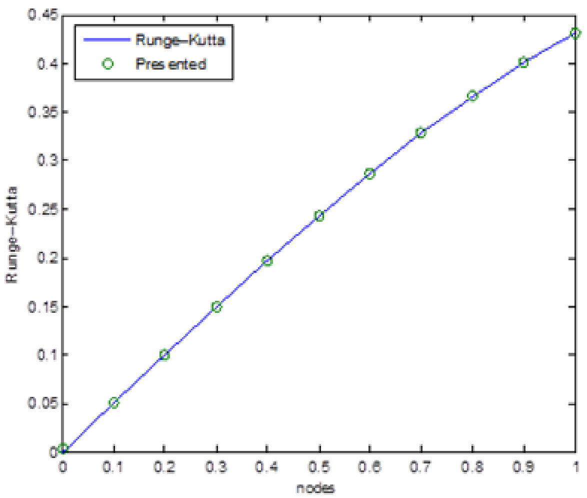

Fig. 2 Comparison of solutions in Example 1

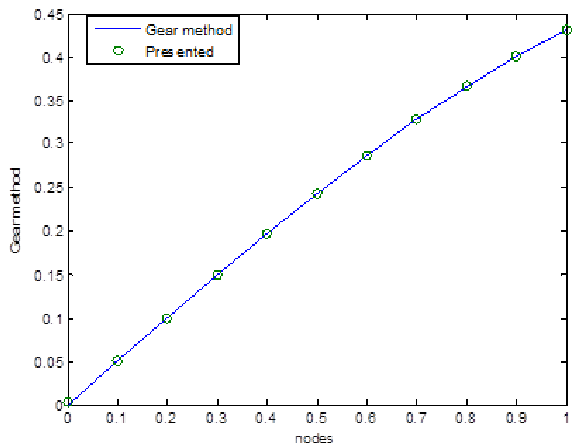

Fig. 3 Comparison of solutions in Example 1

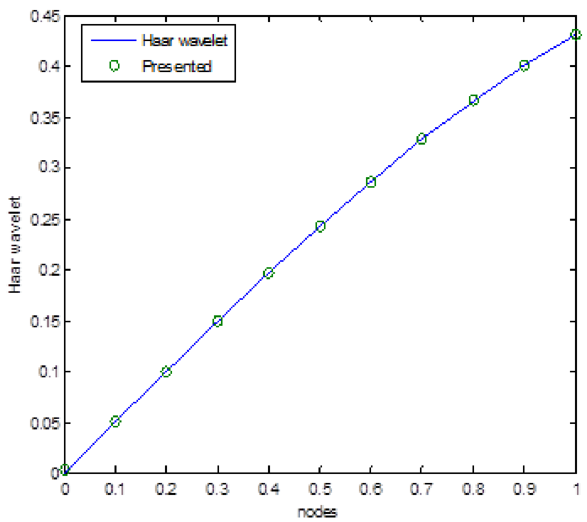

Fig. 4 Comparison of solutions in Example 1

Example 2: Consider the Blasius equation [9]:

$$2y'''(t) + y''(t)y(t) = 0$$

with initial conditions $y(0) = 0$, $y'(0) = 0$, and $y''(0) = A$.

The results of our approach are well accepted (see Table 2) as Bernstein polynomials method [9] for the constant $A = 1$.

Table 2 Numerical results of Example 2 for A = 1

t	Hybrid block method [26]	Bernstein polynomials [9]	Presented method
0.1	0.004999984	0.004999957	0.0050044
0.2	0.019998673	0.019998669	0.0200000
0.3	0.044998466	0.044989878	0.0451174
0.4	0.079991485	0.079957376	0.0801155
0.5	0.124967444	0.124870055	0.1250022
0.6	0.179902828	0.179677139	0.1811330
0.7	0.244755081	0.244303619	0.2450008
0.8	0.319454511	0.318646010	0.3201233
0.9	0.403894885	0.402568622	0.4050011
1	0.497922474	0.495900385	0.5001122

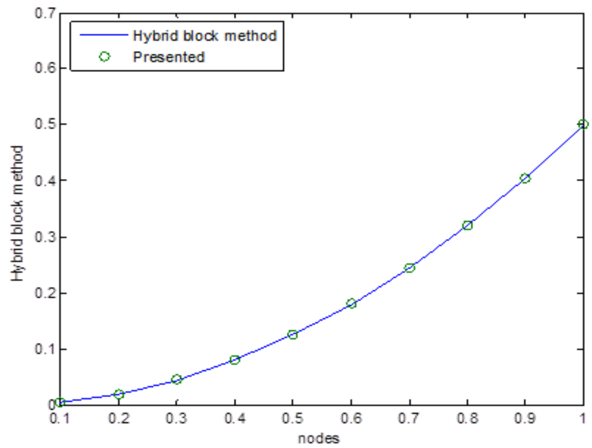

Fig. 5 Comparison of solutions in Example 2

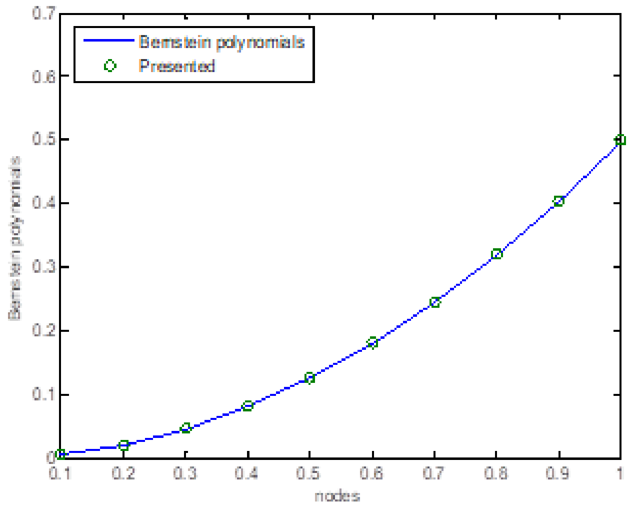

Fig. 6 Comparison of solutions in Example 2

Example 3: The Jerk equation contains displacement times, velocity-time acceleration, and velocity.

$$y'''(t) + y'(t) - y(t)y'(t)y''(t) = 0$$

with initial conditions $y(0) = 0$, $y'(0) = B$, and

$y''(0) = 0$.

A comparison of absolute errors given by the modified differential transform method (MDTM) and our approach can be seen in Table 3 for a constant value of $B = 0.2$. The report shows that our technic gives better results than MDTM.

Table 3 Comparison of absolute errors (b = 0.2) of Example 3

t	MDTM [2/2], [3/3]	MDTM [4/4], [5/5]	Presented method
0.125	8.7000E−08	1.6179E−05	1.7261E−08
0.25	9.8000E−08	3.0115E−05	8.9490E−08
0.375	9.4500E−07	4.0301E−05	1.4337E−07
0.5	3.6420E−06	4.4559E−05	1.1351E−07
0.625	1.1433E−05	4.3207E−05	1.7607E−08
0.75	2.7201E−05	3.5691E−05	8.8824E−08
0.875	5.5915E−05	2.3357E−05	1.4370E−07
1	1.0285E−04	7.9020E−06	1.1418E−07
1.125	1.7268E−04	9.2310E−06	1.7252E−08
1.25	2.7086E−04	2.5171E−05	9.0337E−08
1.375	4.0002E−04	3.8803E−05	1.5529E−07
1.5	5.6231E−04	4.8233E−05	1.8287E−07
1.625	7.5697E−04	5.2924E−05	2.4778E−07
1.75	9.8079E−04	5.3130E−05	3.5759E−07
1.875	1.2287E−03	4.9405E−05	5.2560E−09
2	1.4929E−03	4.3484E−05	2.8260E−06
0.25	9.8000E−08	3.0115E−05	8.9490E−08

Example 4: The following unforced Duffing oscillator that represents the free vibration of the pendulum is

$$y''(t) + y(t) + \varepsilon y^3(t) = 0$$

with prescribed initial conditions $y(0) = A$, $y'(0) = B$.

The exact solution to this equation is not available. Therefore, we compare our results with available methods. Finally, we discuss a few cases for different parameters ε, A, and B values.

Case-I: Take $\varepsilon = 0.3$, $A = 0$, $B = 1$. Figure 7 visualizes compared solutions at different values of t with the modified variational iteration method (MVIM) given by [11]:

$$y(t) = 0.500407\cos(0.990604t) - 0.000406818\cos(2.95275t).$$

Case-II: In this case, we have $\varepsilon = -0.1$, $A = 0.5$, $B = 0$.

Figure 2 shows that the solution is in good agreement with the modified differential transform method (MDTM)

$$y(t) = 0.928746\sin(1.10982t) - 0.0103828\sin(2.96113t),$$

as visualized in Figure 8.

Case-III: Consider $\varepsilon = -1/6$, $A = 0$, $B = 1.6376$. Figure 9 presents comparison of solutions with trigonometric solution $y(t) = 2.058\sin(0.7t) + 0.0816\sin(2.1t) + 0.0337\sin(3.5t)$.

In all the above cases, it is noticed that the numerical solutions achieved by our method coincide quite well with other methods available in the literature and signify that the proposed method is viable and convergent.

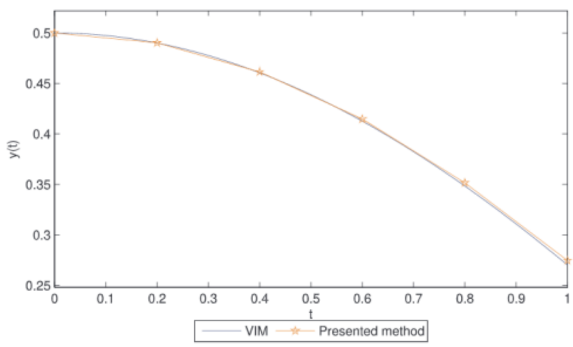

Fig. 7 Comparison of solutions (case-I) in Example 4

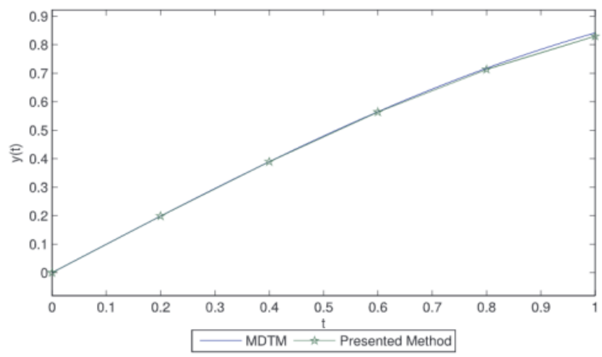

Fig. 8 Comparison of solutions (case-II) in Example 4

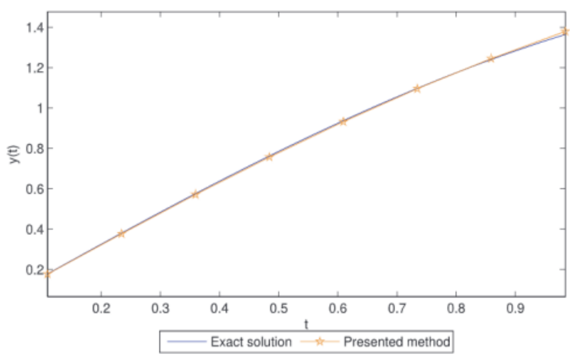

Fig. 9 Comparison of solutions (case-III) in Example 4

5. Conclusion

In this paper, an approximate method for solving nonlinear differential equations was determined by a new approximation system based on inverse Laplace transforms using Chebyshev polynomials functional matrix of integration. In the most general form has been proposed and investigated. For this purpose, the presented method is based on the inverse Laplace transforms using Chebyshev polynomials functional matrix of integration to find the approximate solution. A comparison with the exact solution reveals that the presented method is very effective and convenient. Furthermore, the numerical results show that the accuracy improves, hence better results. Also, from the obtained approximate solution, we conclude that the proposed method gives the solution in excellent agreement with the exact solution.

In this work, the Laplace Adomian decomposition method is modified using the numerical inverse Laplace transform approach grounded upon Bernoulli's functional integration matrix. Moreover, the Laplace Adomian decomposition method is a process that offers a recurrence relationship since operational matrices typically transform differential equations into a system of algebraic equations. Therefore, these two concepts were combined and estimated the efficacity and connection of the approach to some nonlinear differential equations similar to Duffing equation, Van - der Pol equation, Jerk equation, and Blasius equation. The main advantage of the modified technique is that various nonlinear differential equations are answered. The results were compared with other available methods like Haar wavelet, algebraic method, Adam Bashforth multi-step approach, two-point block technic, hybrid block method, Bernstein polynomials, variation replication, modified differential transform, etc., which shows the validity and accuracy of our procedure.

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